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Informing Grades 1–6 Mathematics Standards Development:

What Can Be Learned From High-Performing Hong Kong, Korea, and Singapore?

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Executive Summary

The United States is embarking on a historic policy reversal as it moves toward developing common education standards in reading and mathematics. Supporting this movement is the U.S. Department of Education's $4.3 billion Race-to-the-Top (RttT) competition under the American Recovery and Reinvestment Act (ARRA) and the Common Core State Standards Initiative sponsored by the National Governors Association and the Council of Chief State School Officers. To inform these efforts, this report examines one approach to internationally benchmarking mathematics standards for grades 1–6 against the composite standards of three high-mathematics-performing Asian countries: Hong Kong, Korea, and Singapore.

The U.S. common standards movement offers a unique opportunity to address a well-documented weakness found in many State mathematics standards: many topics are taught in a single grade and many topics are repeated over several grades. This topic spread has led to the well-known characterization of U.S. elementary mathematics curriculum expectations as "a mile wide and an inch deep" (Schmidt, Houang, & Cogan, 2002).

The move to common internationally benchmarked standards offers an opportunity to model U.S. standards off of those of high-performing countries such as Singapore, which offer a more coherent and focused set of expectations. The composite Hong Kong, Korea and Singapore standards developed in this report present one effort to internationally benchmark grades 1–6 mathematics standards against high-performing nations.

Methodology

The Hong Kong, Korean, and Singapore standards were chosen for international benchmarking because of their high performance on the Trends in International Mathematics and Science Study (TIMSS) assessments and the availability of these standards in English on the Asian Pacific Economic Cooperation (APEC) website. Because of the concern over lack of rich mathematical content progression with many U.S. state standards, our particular focus in developing a composite set of standards is on learning progressions—how systematically the mathematical content progresses across the grades within a broad mathematical topic.

The methodology for developing the composite standards that are focused on learning progressions proceeded along the following steps:

1. Transform the Asian standards from their published organization of mathematical content by grade across mathematics strand (e.g., numbers) into an organization of mathematical content by mathematical strand across grades. The across-grade organization of topics focuses on the development of mathematical content within a topic.

2. Identify the core mathematics topics for each strand. These topics are generally apparent from the topic structure of the standards.

3. For each core topic, identify the learning progressions across grades for that topic from each standard. A learning progression represents "a sequence of successively more complex ways of reasoning about a set of ideas" (National Assessment Governing Board, 2008).

4. Create the composite standards for each topic that most clearly indicate the learning progression for each topic over the grades. The following rules were employed in creating the composite standards:

    - If core topic contents were similar for the three standards grade by grade, the content of the standard that was judged to offer the clearest learning progression was chosen.
    - If core topic contents were similar overall across standards but differed in some respects grade by grade, the composite standard reflected a judgment as to which standard offered the most in-depth or clearest learning progression.
    - If core topic contents differed on some competency that was not in the other standards, that competency was included only if it added to and was consistent with the overall learning progression.

Findings

The composite standards have a number of features that can inform an international benchmarking process for the development of K–6 mathematics standards in the United States.

First, the composite standards *concentrate the early learning of mathematics on the numbers, measurement, and geometry strands* with less emphasis on data analysis and little exposure to algebra. The Hong Kong standards for grades 1–3 devote approximately half the targeted time to numbers and almost all the time remaining to geometry and measurement.

Second, the composite standards *sequence topics* within strands to support in-depth and efficient development of mathematics content following a logical development of mathematical knowledge. For example, the numbers strand sequence progression is whole numbers, arithmetic operations, fractions, decimals, ratios, and percents. Measurement introduces linear measurement followed by perimeter and area (two-dimensional measurement) and then the more-complicated volume (three-dimensional measurement). Geometry initially introduces the features of shapes, proceeds to cover two-dimensional geometry along with angles and parallel lines, and concludes with the features of three-dimensional figures. Data analysis starts with pictograms, a visual and more-familiar way to examine data, and then moves on to bar charts and more-complicated continuous line charts.

Third, the composite standards *sequence mathematical competencies within a topic* across the grades according to a mathematically logical progression. Several illustrations occur within the numbers strand. Whole numbers are ordered by size, with grade 1 addressing numbers up to 100, grade 2 up to 1,000, grade 3 up to 10,000, and grade 4 up to 100,000. Grade 5 emphasizes an understanding of large numbers in general. Multiplication is also carefully developed, with grade 2 starting with the basic multiplication concept and multiplication tables for 2, 3, 4, 5, and 10; grade 3 extends to tables 6, 7, 8, and 9 along with multiplication of one digit by two and three digits; grade 4 introduces associative and commutative properties and multiplication of two-digit

numbers by three-digit numbers; and grade 5 covers common multiples and the relation with common divisors.

Fourth, the *ordering of content for one topic is frequently aligned to reinforce the content of another topic* for the same or prior grades. Linear measurement in grade 1 introduces the centimeter, which is aligned with grade 1 exposure to whole numbers up to 100. Grade 3 introduces kilometers and millimeters after 1,000 is taught within the whole-numbers strand of grade 2. Grade 3 introduces the multiplication and division of money (e.g., relations between total costs with price and quantity), thus reinforcing the learning of multiplication and division in grades 2 and 3. Still another example of cross-topic reinforcement occurs within grade 6 data analysis, which introduces pie charts around the same time that circles are introduced in geometry.

In addition, it is important to note that in many cases, particularly within the number strand, the composite standards show a grade placement of a particular skill or concept that is one year earlier than is common in much of the United States. Although this finding is notable, we believe that it is the coherent learning progressions and content connections that are much more important to emulate than the grade placement of particular topics. Further, the delineation of content by learning progressions facilitates an adjustment of the grade placement of content to fit the learning pace of individual students within a common standards framework that all students are eventually expected to master.

In conclusion, the development of standards is only the front end of a long-term reform process. It is critical that sound standards be written to guide the rest of the reform process, however, we do not imply that merely replicating these composite standards is sufficient. Rather we have presented a set of composite mathematics standards of the three Asian high performers to offer a theoretically and empirically valid international benchmark for the development of common U.S. standards in mathematics. We have explained how these composite standards differ from the typical curriculum found in the United States and noted how they address some of the recognized deficiencies found in the U.S. grades 1–6 mathematics curricula.

Informing Grades 1–6 Mathematics Standards Development: What Can Be Learned From High-Performing Hong Kong, Korea, and Singapore?

The United States is embarking on a historic policy reversal as it moves toward developing common education standards in reading and mathematics. One of the most prominent forces supporting this movement is the U.S. Department of Education's $4.3 billion Race-to-the-Top (RttT) competition under the American Recovery and Reinvestment Act (ARRA). RttT encourages consortia of states to develop proposals to address four major education priorities, one of which is developing "common, high-quality K–12 standards that are internationally benchmarked."

To date, 48 states have signed on to the Common Core State Standards Initiative, sponsored by the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO). This initiative calls for developing state standards against international benchmarks.

> By signing on to the Common Core State Standards Initiative, governors and state commissioners of education across the country are committing to joining a state-led process to develop a common core of state standards in English-language arts and mathematics for grades K–12. These standards will be research and evidence-based, internationally benchmarked, aligned with college and work expectations and include rigorous content and skills. (NGA Center & CCSSO, 2009)

To inform this process, this study examines one approach to internationally benchmarking mathematics standards for grades 1–6. The common nature of mathematics across countries makes mathematics a particularly appropriate subject for international benchmarking comparisons. This study compares the standards for Hong Kong, Korea, and Singapore, three of the highest-performing countries on the Trends in International Mathematics and Science Study (TIMSS) in grades 4 and 8, and uses these three sets of standards to create a single composite set of standards. The standards from these three countries are available in English on the Education Network (EDNET) of the Asian Pacific Economic Cooperation (APEC, n.d.).

A shift in the United States to a common set of mathematics standards across states represents a unique opportunity to address a well-documented weakness of many state mathematics standards: the same topic is often repeated over several grades instead of building depth of understanding as students process through the grades (Reys, 2006). This topic spread has led to the well-known characterization of U.S. elementary mathematics curriculum expectations as "a mile wide and an inch deep" (Schmidt, Houang, & Cogan, 2002).

The tendency of U.S. state standards to repeat topics has been carefully documented in a comparison of the Singapore mathematics standards, one of the world's best performers on international mathematics assessments, with the standards from the seven largest U.S. states. The median coverage of the seven state standards was 70% more topics per grade than Singapore. Florida covered an average of 160% more topics per grade (Ginsburg, Leinwand, Anstrom, & Pollock, 2005). With so many more topics per grade, U.S. standards are not able to expose students to the same depth of content as Singapore standards.

The movement at federal and state levels toward common internationally benchmarked mathematics standards offers a rare opportunity to strengthen the focus and the depth of U.S. mathematics standards in ways that more closely reflect the standards of the high international performers, such as Singapore. The question at hand is how best to describe the international standards against which the United States should benchmark.

This study has developed a set of composite international mathematics standards based on the elementary mathematics standards of Hong Kong, Korea, and Singapore—three high-performing countries. The composite standards are organized by five mathematical strands—numbers, measurement, geometry, data analysis, and algebra—that are used by the National Council of Teachers of Mathematics (NCTM). Additionally, these composite standards cover introductory mathematics in grades 1–6 because international comparisons show that a country's performance in the elementary grades has a strong influence on its performance on international assessments in the later grades (Ginsburg, Cooke, Leinwand, Noell, & Pollock, 2005).

Within each strand, the composite set of standards is organized by major mathematical topics. For each topic, *learning progressions of specific competencies within each topic across grades* are presented. For example, the fractions topic contains a grade-by-grade learning progression of fraction concepts that builds from fraction of a whole to equivalent fractions to improper fractions and mixed numbers.

The case for incorporating learning progressions into the common standards development initiative based on international experience was set out in an *Education Week* commentary (Fuhrman, Resnick, & Shepard, 2009):

> Countries such as Singapore, Japan, South Korea, and the Czech Republic that score above the United States in international assessments of science and math provide teachers with much clearer guidance on the key ideas to be explored and mastered in each grade. … Well-developed curriculum sequences ensure that students take on more-advanced topics as they progress through school, thus building a deep understanding of how the topics relate to one another. … Scientists who study learning have gained insights about cognitive development that suggest the importance of building curricula based on sequences, or progressions, of increasingly sophisticated concepts and knowledge applications.

For comparison purposes, for each strand, NCTM's curriculum focal points are also presented.

> Curriculum focal points are important mathematical topics for each grade level, pre-K–8. These areas of instructional emphasis can serve as organizing structures for curriculum design and instruction at and across grade levels. (NCTM, 2006, p. 5)

In essence, the focal points identify core content by strand for each grade.

After an introduction to the study's methodology, the following sections describe this set of composite internationally benchmarked standards and illustrate the learning progressions that emerge across grades 1–6. The topic development is organized around the major mathematics strands. The final section presents the implications of the study for the development of common standards for the United States.

- Section I. Methodology
- Section II. Numbers and Operations
- Section III. Measurement
- Section IV. Geometry
- Section V. Data Analysis
- Section VI. Algebra
- Section VII. Implications for U.S. Standards Development

I. Methodology

This analysis develops and describes a single composite set of mathematics standards for grades 1–6 that are based on the mathematics standards of Hong Kong, Korea, and Singapore. This methodology section explains the basis for the selection of these three Asian countries' standards for international benchmarking and the process for developing the composite standards. The process includes topic selection within strands and the construction of learning progressions of skills and concepts within each topic.

Hong Kong, Korea, and Singapore were chosen for the benchmarking exercise because they represent three of the highest performing countries on three international mathematics assessments (see Table 1). Korea and Singapore were among the top three countries on all tests taken. Hong Kong was also always near the top, placing first and fourth on TIMSS and third on the Program for International Student Assessment (PISA).

Table 1. Rankings of Hong Kong, Korea, and Singapore on TIMSS Grade 4 and Grade 8 and PISA–Age 15

	Rank (36 countries) TIMSS — Grade 4 (2007)	Rank (49 Countries) TIMSS — Grade 8 (2007)	Rank (57 Countries) PISA — Age15 (2006)
Hong Kong	1	4	3
Korea	—	2	4
Singapore	2	3	—
Source: Mullis, Martin, & Foy (2008) and Organization for Economic Co-Operation and Development (OECD) (2007).			

The mathematics standards for the three countries are available at http://hrd.apec.org, the website of the Human Resource Development (HRD) working group of the Asian Pacific Economic Cooperation (APEC), an organization composed of 21 countries bordering the Pacific Ocean. The standards from all three countries organize content into broad mathematical strands that cover numbers, measurement, geometry, data analysis, and algebra. None of the three sets of grades 1–6 standards includes probability until after grade 6, and although none of the three countries stresses content in the algebra strand at the elementary level, algebra is interwoven through the properties of operations, patterns, and representing and solving equations.

Analyses of state or country standards typically focus on the topical content for each grade but rarely attend to the cumulative development of topics over grades. For example, one well-known study (Reys, Dingman, McNaught, Regis, & Togashi, 2006) identified the need for common standards by showing that particular mathematical topics were covered at widely differing grades

across the different U.S. states. However, the study did not explicitly examine the cumulative development of mathematics topics across grades (i.e., learning progressions) that is a central feature of the standards from the high-performing Asian countries.

In contrast, the composite mathematics standards developed in this study are organized around major mathematics topics within each strand and illustrate the development of mathematical competence in subtopics across grades. The Singapore mathematics standards refer to this progression of content development as a spiral approach:

> Using a spiral design of the curriculum, each topic is revisited and introduced in increasing depth from one level to the next. This enables students to consolidate the concepts and skills learned and then further develop them. (Singapore Ministry of Education, 2006)

The development of the composite Hong Kong, Korea, and Singapore mathematics standards was conducted in three steps: (1) identify the *core mathematics topics* taught within each strand across countries; (2) identify each country's grade-by-grade *sequence of mathematical competencies* for each core topic; and (3) create *composite standards* for each core topic by drawing from the learning progressions in the standards of each of the three countries.

Identify the core mathematics topics for each strand. These topics are generally apparent from the topic structure of the standards. The Korean standards present a summary table of "Content Organization" that identifies the major topics. The Hong Kong and Singapore standards explicitly identify the topics and subtopics associated with each strand by grade. Table 2 shows the core mathematics topics identified within each strand based on the three sets of standards.

Table 2. Core Mathematics Topics: Grades 1–6

Numbers	Measurement	Geometry	Data/Probability	Patterns/Algebra
• Whole numbers • Addition/ subtraction • Multiplication/ division • Fractions • Decimals • Ratios • Percents	• Linear measurement • Perimeter/area • Volume • Nongeometric measurement – *Time (clock)* – *Time (calendar)* – *Money* – *Weight*	• 2-D shapes • 3-D shapes • Lines • Angles	• Classification of objects by attributes into groups • Pictograms • Bar graphs • Tables • Line graphs • Averages • Pie charts	• Symbols • Equations

For each core topic, identify the learning progressions across grades for that topic from each standard. This underlying concept of learning progressions has been described by the National Assessment of Educational Progress (NAEP), as follows:

> A learning progression is a sequence of successively more complex ways of reasoning about a set of ideas. … In other words, the progression from novice learner to competent learner to expert begins with the acquisition of relevant experiences, principles, concepts, facts, and skills and moves to the accumulation and organization of knowledge in a

specific domain and finally to expertise *after extensive experience and practice.* (National Assessment Governing Board, 2008)

The learning progressions for each country's standards by core topic were constructed by pulling out the content for that core topic grade by grade. For example, the content of the linear measurement topic was identified for each grade by country.

Create the composite standards. First, the grades covered for each topic were compared across the three sets of standards to ensure similarity. For most topics, the three sets of standards cover the topics over similar grades, although exceptions often occurred at the end points. An illustration of a typical grade pattern is the "measurement of length" presented in Table 3. All three sets of national standards cover length measures in grades 1–3 and Singapore covers measurement of length in grade 5 as a learning objective connected with showing an application of decimals. An outlier grade (e.g., grade 5 Singapore for measurement of length) was included in the composite standard if it was judged to add important content to the learning progression.

Table 3. Measurement of Length by Grade: Hong Kong (H), Korea (K), and Singapore (S)

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Length	H, K, S	H, K, S	H, K, S		S	

The composite standards for each core mathematics topic were then created by consolidating the competencies from the three sets of standards that describe what students should know and be able to do across the grades. The composite standards represent a judgment designed to present the essential learning competencies from the three sets of standards that most clearly indicate the learning progression for each topic over the grades. The following rules were employed in creating the composite standards:

- If core topic contents were similar for the three standards grade by grade, the content of the standard that was judged to offer the clearest learning progression was chosen.
- If core topic contents were similar overall across standards but differed in some respects grade by grade, the composite standard reflected a judgment as to which standard offered the most in-depth or clearest learning progression.
- If a core topic contents differed on some competency that was not in the other standards, that competency was included only if it added to and was consistent with the learning progression.

Table 4 shows a sample learning progression for the topic of length within the measurement strand, with some key features:

- The learning progression contains two broad sequences or subtopics, one for concept of length and a second for tools/measuring length.
- The concept of length sequence introduces the concept in grade 1 along with the centimeter unit and proceeds in later grades to present different sizes of measuring units and decimals.
- The tools/measuring length sequence presents similar skills at each of grades 1, 2, and 3, but with a clear progression in the selection of the measurement units used.

Note that the composite standards represent the progression of core learning objectives for a topic and may omit some of the details associated with the standards for a particular topic from any of the three countries.

Table 4. Composite Learning Progression for Length Within Measurement

Measurement Topic	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Concept of length	• Understand concepts of length and distance • Understand long, longer, longest, short, shorter, shortest • Understand centimeter	• Understand that a meter is greater than a centimeter	• Understand that a kilometer is greater than a meter and that a millimeter is smaller than a centimeter • Convert compound units to a smaller or a larger unit		• Convert from a smaller unit to a larger unit and vice versa in decimal form	
Tools/measuring length	• Measure and compare lengths of objects and distance with centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in meters and centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in kilometers and millimeters • Estimate lengths and distances • Measure length with appropriate tools			

The composite standards for each strand are also compared with the NCTM Curriculum Focal Points for that strand. As noted, the development of focal points was an effort by NCTM to address the "mile wide inch deep" criticism of U.S. state standards as well as an effort to make state standards more uniform. It bears repeating that the focal points are designed as benchmarks to guide a more coherent, and less repetitive, curriculum. They are not intended to provide either the detail or the learning progressions that are found in a complete set of standards. They do, however, serve as a useful check against which to examine the composite standards.

The following sections describe the results of applying these steps to develop the composite standards for the core mathematical topics within the numbers, measurement, geometry, data analysis, and algebra strands.

II. Numbers and Operations

Numbers form the basic building blocks for all early learning of mathematics. Quite appropriately, the numbers strand receives the greatest attention among the five strands at the elementary grades. As an illustration (see Table 5), the Hong Kong standards, which explicitly suggest classroom time allocations for different strands, recommend that more than 40% of total time be allocated to numbers across grades 1–6, more than twice the amount for any other strand.

Table 5. Hong Kong's Suggested Time Allocations by Mathematics Strand: Primary 1–3 and 4–6

Key Mathematics Stages	Numbers	Geometry	Measurement	Data Analysis	Algebra	Spare Periods
Primary 1–3	46%	15%	20%	3%	0%	16%
Primary 4–6	41%	13%	16%	10%	7%	13%

Composite Standards

The major topics in the numbers strand for all three countries follow a similar pattern across grades that is dictated by the logic of mathematics learning. The standards first introduce whole numbers and place value, starting with numbers to 100. The presentation of whole numbers continues over the grades, with students systematically exposed to a larger number of digits. The standards simultaneously introduce arithmetic operations of addition and subtraction followed by multiplication and division, with the number of digits of arithmetic problems expanding in tandem with students' exposure to whole numbers and place value. Fractions, decimals, and percentages follow once students have a firm grasp of basic whole number concepts and operations.

Whole numbers. Counting is one of the first skills that children learn in mathematics. Along with rote, sequential counting, children learn to count and compare the number of objects in sets. Counting progresses with larger numbers over the grades as place value concepts are learned. With larger numbers, children apply their place value understanding to comparing, ordering, and rounding numbers. Numbers are further differentiated into odd and even groups.

Table 6 shows the composite set of standards for whole numbers.

- Grade 1 addresses whole numbers up to 100. Basic whole-number skills include counting the numbers of objects in a set, which requires one-to-one correspondence between the objects and the number; comparing the size of sets; ordering numbers; and knowing that numbers show position (1st, 2nd). Place value is introduced to distinguish tens and ones, and the correspondence between numeral symbols and words is taught.
- The grade 2 learning progression extends recognizing and ordering whole numbers up to 1,000; understanding place value of hundreds, tens, and ones; and adding 10 and 100 to numbers mentally.
- Grade 3 focuses on numbers up to 10,000 with place value to thousands.
- At grade 4, numbers are up to 100,000, and rounding and approximation are introduced.
- Grade 5 explicitly treats understanding of large numbers up to a hundred million along with the concepts of approximation, estimation, and rounding.

Table 6. Composite Standards: Numbers—Whole Numbers

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Whole numbers to 100: • Count to tell the number of objects in a given set • Count forward and backward • Compare the number of objects in two or more sets • Use ordinal numbers (first, second, up to tenth) and symbols (1st, 2nd, 3rd, etc.) • Use number notation and place values (tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers	Whole numbers to 1,000: • Count in tens and hundreds • Use number notation and place values (hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers	Whole numbers to 10,000: • Use number notation and place values (thousands, hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers • Understand odd and even numbers	Whole numbers to 100,000: • Use number notation and place values (ten thousands, thousands, hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers • Round numbers to the nearest 10 or 100	• Develop an understanding of large numbers • Develop the concept of approximation • Estimate the number of a large quantity of objects • Round large numbers in thousands, ten thousands, hundred thousands, millions, ten millions, hundred millions	

Addition and subtraction. The number standards present addition along with subtraction so that students understand the meanings and relationship. The standards require mastering adding and subtracting smaller numbers, thereby promoting automaticity. The learning progression builds up addition and subtraction skills by introducing successively larger numbers over grades. Word problems are also introduced early on as a way of promoting students' understanding of addition and subtraction concepts (Har & Hoe, 2007). Table 7 shows the composite set of standards for addition and subtraction.

- The concepts of addition and subtraction are introduced in grade 1. Addition and subtraction are initially constrained within 20 and includes learning all the different combinations of sums through 9 plus 9 and finding an unknown number within the combination. By the end of the year, addition and subtraction expand to sums and differences within 100 without regrouping and the introduction of 1-step word problems. Addition and subtraction facts are taught in all three countries using what Singapore calls "number bonds" and the United States calls "fact families" that relate, for example, 4 + 7, 7 + 4, 11 – 4, and 11 – 7 to promote understanding and reduce memory load.
- In grade 2, addition and subtraction are extended to numbers involving three digits, 2-step word problems, and mental calculation.

- In grade 3, addition and subtraction are extended to numbers up to four digits, and again there is a stress on word problems.

Table 7. Composite Standards: Numbers—Addition and Subtraction

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Addition and subtraction: • Understand situations for, and the meaning of, addition and subtraction • Use the addition symbol (+) or the subtraction symbol (−) • Compare two numbers within 20 to tell how much one number is greater (or smaller) than the other • Recognize the relationship between addition and subtraction • Build the addition bonds up to 9 + 9 • Solve 1-step word problems involving addition and subtraction within 20 • Add more than two 1-digit numbers • Add and subtract within 100 without regrouping involving – *a 2-digit number and ones* – *a 2-digit number and tens* – *two 2-digit numbers* • Use mental calculation for addition and subtraction – *within 20* – *involving a 2-digit number and ones without renaming* – *involving a 2-digit number and tens*	Addition and subtraction of numbers up to three digits: • Solve up to 2-step word problems involving addition and subtraction • Use mental calculation for addition and subtraction involving – *a 3-digit number and ones* – *a 3-digit number and tens* – *a 3-digit number and hundreds*	Addition and subtraction of numbers up to four digits: • Use the terms "sum" and "difference" • Solve up to 2-step word problems involving addition and subtraction			

Multiplication and division. The concept of multiplication can be expressed as repeated addition or represented as the total in a rectangular array of objects. Division can be interpreted as repeated subtraction or as either the number of groups formed when a set of objects is shared equally or the number of objects in each group when objects are grouped into a given number of groups.

The complexity of the multiplication and division builds over the grades, beginning with the meaning of multiplication and the multiplication facts with factors of 2, 3, 4, 5, and 10 in grade 2 and with factors of 6, 7, 8, and 9 in grade 3. In the upper grades, the learning progression

increases the numbers of digits, formally introduces the associative and commutative properties of multiplication, and promotes an understanding of the common and greatest common divisors. As with addition and subtraction, multiplication and division facts are taught in all three countries using what Singapore calls number bonds (and the United States calls fact families) that relate, for example, 4×7, 7×4, $28 \div 4$, and $28 \div 7$ to promote understanding and reduce memory load. Table 8 shows the composite set of standards for multiplication and division.

- In grade 2, students are introduced to the concept of multiplication and build the multiplication tables by mastering the easier facts with factors of 2, 3, 4, 5, and 10 and the commutative property of numbers. The basic idea of division is introduced as sharing items in a set to form equal-size groups, and the standards support understanding division as sharing when given the number of items in a set or the number of sets. Concept development is reinforced through 1-step word problems involving multiplication and division.
- In grade 3, students encounter the remainder of the multiplication facts through 10 and are introduced to one-digit multipliers with two- and three-digit numbers. Also in grade 3, students encounter formal division procedures for dividing two- and three-digit dividends by one-digit divisors. Two-step word problems involving the four operations reinforce conceptual understanding and connections across operations.
- Grade 4 introduces the associative and commutative properties of multiplication and increases the number of digits involved in multiplication and division problems. Word problems may involve 3 steps.
- Grade 5 introduces the ideas of and relationships between greatest common divisor and least common multiple. The order of mixed operations including brackets is presented.

Table 8. Composite Standards: Numbers—Multiplication and Division

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	Basic multiplication (basic concept and computation): • Understand the situations for, and meaning of, multiplication • Build up the multiplication tables of 2, 3, 4, 5, and 10 • Discover the commutative property of multiplication through concrete examples (e.g., 2 × 3 = 3 × 2) Basic division (basic concept and computation): • Develop the concept of division: sharing and grouping • Divide a quantity (not greater than 20) into equal sets given – *the number of objects in each set* – *the number of sets* • Recognize the relationship between multiplication and division • Solve 1-step word problems involving multiplication and division within the multiplication tables	Multiplication: • Build up the multiplication tables of 6, 7, 8, and 9 • Perform multiplication with a multiplier of 1 digit and a multiplicand of 2 or 3 digits Division: • Understand the situations for, and meaning of, division • Perform basic division by short division • Perform division with a divisor of 1 digit and a dividend of 2 or 3 digits with and without remainders • Use the terms "product," "quotient," and "remainder" • Solve up to 2-step word problems involving the four operations, including estimating answers	Multiplication: • Discover the associative property of multiplication through concrete examples • Apply the commutative and associative properties of multiplication in computation (e.g., 2 × 8 × 5 = (2 × 5) × 8) • Perform multiplication with a multiplier of 2 digits and a multiplicand of 2 digits and then 3 digits Division: • Perform division with a divisor of 2 digits and a dividend of 2 and then 3 digits • Recognize divisibility when the divisors are 2, 5, and 10 • Identify 1-digit factors of 2-digit numbers • Distinguish between factors and multiples • Solve up to 3-step word problems involving the four operations, including estimating answers	Divisors and multiples: • Understand the meaning of "divisor," "common divisor," and "greatest common divisor" and know how to solve for them • Understand the meaning of "multiple," "common multiple," and "least common multiple" and know how to solve for them • Understand the relation between divisors and multiples and know how to apply them • Multiply and divide by 10, 100, and 1000 mentally • Use order of operations, combined operations involving the four operations, and brackets • Solve word problems involving the four operations, including estimating answers	

Fractions. The topics for fractions follow a similar learning progression across all three sets of standards. The standards introduce the concept of a fraction as part of a whole. Students next order fractions with like denominators. Equivalent fractions follow, as students learn different representations for fractions with the same value, along with simplifying fractions. Operations on fractions are initially limited to addition and subtraction. Multiplication and division of fractions follow with several competencies focused on explaining the complicated concept of division involving fractions. Mixed calculations of both fractions and decimals are also covered.

Although students commonly deal with fractions every day, they fall prey to a range of common misconceptions. For example, students mechanically form a fraction by counting parts regardless of whether the parts are of equal size. Other problems arise with the idea that the multiplication

of two fractions can lead to a smaller number, particularly when multiplication is defined as repeated addition. Similarly, if division is dividing into equal groups, how can division of fractions produce a larger number? Rules about computing with fractions also often cause difficulties. The standards need to present concepts clearly and support instruction that avoids and corrects student misconceptions. Table 9 shows the composite set of standards for fractions.

- **Fraction concepts.** The concept of a fraction as a part of a whole and the comparison and ordering of like fractions with denominators not exceeding 12 are introduced in grade 2 as students are simultaneously learning to count. In grade 3, these early notions of fractions are extended to equivalent fractions with practice listing equivalent fractions and writing an equivalent of a fraction given the numerator or denominator. Ordering is also extended to comparing and ordering unlike fractions. Grade 4 broadens the idea of fractions to cover mixed numbers and improper fractions.
- **Fraction arithmetic operations.** Grade 3 introduces addition and subtraction of like fractions with denominators limited to 12. Grade 4 extends addition and subtraction of like fractions beyond 12 and introduces the multiplication of proper and improper fractions. Grade 5 stretches addition and subtraction to cover addition and subtraction with unlike denominators. Multiplication of fractions involving whole numbers, fractions, and mixed numbers is also presented. Grade 6 is devoted to explaining division of fractions and also includes arithmetic calculations involving both fractions and decimals.

Table 9. Composite Standards: Numbers—Fractions

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Fractions/concepts		Fraction of a whole: • Interpret a fraction as part of a whole • Read and write fractions • Compare and order unit fractions and like fractions (denomina-tors less than or equal to 12)	Equivalent fractions: • Recognize and name equivalent fractions • Write the equivalent fraction of a fraction, given the denominator or the numerator • Express a fraction in its simplest form • Compare and order unlike fractions, including comparing fractions with respect to one half (denominators less than or equal to 12)	Mixed numbers and improper fractions: • Understand the concepts of mixed numbers and improper fractions • Express an improper fraction as a mixed number, and vice versa, and express both in simplest form		

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Fractions/arithmetic operations			Addition and subtraction of two related fractions (one denominator a factor of the other) within one whole (denominators of given fractions should not exceed 12)	Addition and subtraction: • Add and subtract like fractions • Add and subtract related fractions (denominators of given fractions should not exceed 12) Multiplication of a proper or improper fraction and a whole number	Addition and subtraction of fractions with unlike denominators: • Add and subtract fractions with unlike denominators Multiplication and division of fractions: • Multiply proper fractions, improper fractions, mixed numbers and whole numbers by proper fractions, improper fractions, and mixed numbers • Divide fractions by whole numbers and whole numbers by fractions	Division of fractions: • Divide proper fractions by proper fractions Mixed calculations with fraction and decimal: • Know how to solve simple calculations with both fractions and decimals

Decimals. Decimals are taught in all three countries by connecting to and extending place value and fraction concepts. The concept of decimal builds from students' place value knowledge that the value of digits to the left of the decimal point builds by multiples of 10 from ones to tens to hundreds and that, by analogy, the value of digits to the right of the decimal point changes from tenths to hundredths to thousandths, decreasing by a factor of 1/10 each place. Along with place value, the number line and conversions of fractions to decimals and vice versa help students understand decimals. Understanding the meaning and size of decimals is supported through the competency of ordering of decimals of different sizes.

Students learn operationally that decimal computations obey rules associated with the four arithmetic operations. However, as with fractions, some results are counterintuitive. Multiplication of a whole number by a decimal less than 1 does not yield a larger number, and division of a whole number by a decimal less than 1 does not yield a smaller number. Also, multiplication by hundredths yields a smaller, not larger, number than multiplication by tenths,

while multiplication by hundreds yields a larger number than multiplication by 10. Table 10 shows the composite set of standards for decimals.

- At grade 4, decimals are introduced by a stress on conceptual understanding of decimals up to three places, building on knowledge of place values, number lines, and fractions, three ways of displaying decimals. The ordering of decimals requires students to apply their understanding of the place value position of decimals.
- At grade 5, the rules for adding and subtracting decimals are introduced. The rules for multiplying decimals, including by whole numbers and by decimals, is also addressed, as well as real-world applications.
- At grade 6, students encounter the division of decimals developed through daily life examples and the relationship between decimals and fractions.

Table 10. Composite Standards: Numbers—Decimals

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
			Decimals up to three decimal places: • Understand notation and place values (tenths, hundredths, thousandths), including identifying the values of the digits in a decimal • Use the number line to display decimals • Compare and order decimals • Convert a decimal to a fraction • Convert a fraction whose denominator is a factor of 10 or 100 to a decimal • Round decimals to the nearest whole number	Decimal addition and subtraction: • Add and subtract decimals up to two places of decimals and for sums involving at most three operations • Estimate the answers Decimal multiplication: • Develop an understanding of multiplication of decimals through daily life examples • Multiply decimals by whole numbers and by decimals • Estimate the answers	Decimal division: • Develop an understanding of division of decimals through daily life examples • Divide decimals by whole numbers, whole numbers by decimals, and decimals by decimals • Perform mixed operations on decimals for sums involving at most three operations • Estimate the answers Decimal conversion: • Convert decimals to fractions and fractions to decimals • Compare fractions by converting them to decimals • Estimate the answers

Ratio. Ratios are comparisons of quantities. Although often expressed as fractions, they are not numbers and cannot be placed on a number line. Adding to potential confusion, whereas 2/5 cannot be added to 3/4 by adding numerators and adding denominators to get 5/9, in the case of ratios, 2 out of 5 "plus" 3 out of 4 can be "added" to get 5 out of 9. Understanding ratios is a key prerequisite for dealing with proportions (equivalent ratios) and percents (ratios to 100). Table 11 shows the composite set of standards for ratios.

- Grade 5 provides a complete presentation of ratios building from the meaning and representation of ratios to equivalent ratios to solving a full range of problems involving whole number ratios and equivalent ratios.

Table 11. Composite Standards: Numbers—Ratios

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr.6
				Ratio (excludes ratios involving fractions and decimals): • Interpret *a* : *b* and *a* : *b* : *c*, where *a*, *b*, and *c* are whole numbers • Express a ratio in its simplest form • Find the ratio of two or three given quantities • Write equivalent ratios and find the missing term in a pair of equivalent ratios • Solve up to 2-step word problems involving ratio, including finding one quantity given the other quantity and their ratio	

Percents. A percent is the ratio of a number to 100. It is common for this number quantity to be money, with percentage being an interest rate or a price reduction. One challenge is for students to calculate the correct base. For example, if A is *x*% more than B does not mean that B is *x*% less than A. Another challenge is to translate word problems involving percents into mathematical representations. To illustrate, the price of a suit at one store is $200, which is 20% more than the price of the same suit at another store. What is the price of the suit at the second store? Table 12 shows the composite set of standards for percent.

- Grade 6 introduces the basic concept of percentages and makes the connection with the relationship to equivalent decimals and fractions. Problem solving in the application of percentages includes finding a percentage, expressing the value given the percentage of a quantity, and solving discount price problems.

Table 12. Composite Standards: Numbers—Percents

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
					Percents (basic concept; conversion of percentages to decimals or fractions and vice versa): • Recognize percentages through daily life examples • Develop an understanding of percentages • Convert percentages to decimals and vice versa Applications of percents: • Solve simple problems on percentages, including finding percentages expressing the value of a percentage of a quantity applying discounts • Estimate the answers

NCTM Curriculum Focal Points

The Curriculum Focal Points are similar to the mathematics standards in Hong Kong, Korea, and Singapore in their emphasis on the numbers strand, with 12 of the 18 focal points for grades 1–6 including a numbers component. The focal points cover all the major number topics identified in the composite standards (see Table 13). A strength of the focal points is their overall stress on the mathematical competencies that students must have in order to develop a strong foundation in the numbers strand. The focal points emphasize conceptual understanding, multiple representations, and connections. In a number of cases, the stress on understanding and explanations is clearer than in the Asian standards on the same topic.

However, the focal points do not, and were not intended to, provide a careful sequential progression of competencies that builds a solid in-depth foundation of knowledge and skills in that topic. The focal points are limited to three topics per grade and therefore typically describe only the mathematical topics at a single grade deemed to be most important for presenting the content for that topic. As the Singapore standards point out, however, the delineation of learning expectations should be presented in a spiral fashion that builds on prior knowledge in previous grades in a highly ordered way. Thus, the numbers strand focal points can be viewed as establishing critical guideposts for student learning around which the composite standards can be used to develop sequences of mathematical competencies that stress understanding, connections, and representations.

The key points of difference between the focal points and the composite set of standards are described below:

- The focal points that address whole numbers in grades 1 and 2 stress counting, ordering, and understanding place value as multiple powers of 10 for numbers up to 100 in grade 1 and 1,000 in grade 2. However, NCTM differs from the Asian composite standards for whole numbers in that it does not examine ordinal numbers and does not extend whole numbers in successive grades to 10,000 and 100,000 and does not include interpreting and rounding large numbers in later grades.
- The focal points for addition and subtraction in grade 1 emphasize understanding addition and subtraction through objects and number lines and connections with counting, along with using a variety of strategies to problem solve. In grade 2, addition and subtraction include fluency with multidigit and mental arithmetic of basic addition and subtraction facts. Along with fluency with algorithms is a stress on why procedures work.
- The focal points for multiplication and division introduce multiple representations (equal-sized groups, arrays, area models, jumps on number lines) to highlight and interpret the meanings of these operations in grade 3. Also in grade 3, the associative, commutative, and distributive properties are stressed, and ratio and rates are connected with multiplication and division problems. Grade 4 stresses fluency with multidigit whole-number multiplication, stressing why the procedures work, and quick recall of multiplication and related division facts. Grade 5 focuses on division of whole numbers.
- Fractions and decimals are treated together in grades 4, 5, and 6 in the focal points. NCTM suggests introducing decimals as an extension of base ten and then relating fractions to decimals on the number line. Addition and subtraction of fractions (with like and unlike denominators) and decimals are both included in grade 5 where connections are made to the measurement strand. Grade 6 extends decimals and fractions to multiplication and division and reinforces this work with multistep problems, including measurement. By making the immediate connection between decimals and fractions, NCTM offers an interesting alternative to the Asian strategy of first introducing fractions and then decimals. The Asian composite spreads the treatment of fractions and decimals over grades 2–6, which provides a more careful development of these competencies.

Table 13. NCTM Focal Points: Numbers

Grade 1. Number and Operations: Developing an understanding of whole number relationships including grouping in tens and ones. Children compare and order whole numbers (at least to 100) to develop an understanding of and solve problems involving the relative sizes of these numbers. They think of whole numbers between 10 and 100 in terms of groups of tens and ones (especially recognizing the numbers 11 to 19 as 1 group of ten and particular numbers of ones). They understand the sequential order of the counting numbers and their relative magnitudes and represent numbers on a number line.

Number and Operations and Algebra: Developing understandings of addition and subtraction and strategies for basic addition facts and related subtraction facts. Children develop strategies for adding and subtracting whole numbers on the basis of their earlier work with small numbers. They use a variety of models, including discrete objects, length-based models (e.g., lengths of connecting cubes), and number lines, to model "part-whole," "adding to," "taking away from," and "comparing" situations to develop an understanding of the meanings of addition and subtraction and strategies to solve such arithmetic problems. Children understand the connections between counting and the operations of addition and subtraction (e.g., adding two is the same as "counting on" two). They use properties of addition (commutativity and associativity) to add whole numbers, and they create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems involving basic facts. By comparing a variety of solution strategies, children relate addition and subtraction as inverse operations.

Grade 2. Number and Operations: Developing an understanding of the base-ten numeration system and place-value concepts. Children develop an understanding of the base-ten numeration system and place-value concepts (at least to 1000). Their understanding of base-ten numeration includes ideas of counting in units and multiples of hundreds, tens, and ones, as well as a grasp of number relationships, which they demonstrate in a variety of ways, including comparing and ordering numbers. They understand multidigit numbers in terms of place value, recognizing that place-value notation is a shorthand for the sums of multiples of powers of 10 (e.g., 853 as 8 hundreds + 5 tens + 3 ones).

Number and Operations and Algebra: Developing quick recall of addition facts and related subtraction facts and fluency with multidigit addition and subtraction. Children use their understanding of addition to develop quick recall of basic addition facts and related subtraction facts. They solve arithmetic problems by applying their understanding of models of addition and subtraction (such as combining or separating sets or using number lines), relationships and properties of number (such as place value), and properties of addition (commutativity and associativity). Children develop, discuss, and use efficient, accurate, and generalizable methods to add and subtract multidigit whole numbers. They select and apply appropriate methods to estimate sums and differences or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems.

Grade 3. Number and Operations and Algebra: Developing understandings of multiplication and division and strategies for basic multiplication facts and related division facts. Students understand the meanings of multiplication and division of whole numbers through the use of representations (e.g., equal-sized groups, arrays, area models, and equal "jumps" on number lines for multiplication, and successive subtraction, partitioning, and sharing for division). They use properties of addition and multiplication (e.g., commutativity, associativity, and the distributive property) to multiply whole numbers and apply increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving basic facts. By comparing a variety of solution strategies, students relate multiplication and division as inverse operations.

Number and Operations: Connecting ratio and rate to multiplication and division. Students use simple reasoning about multiplication and division to solve ratio and rate problems (e.g., "If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12"). By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative sizes of quantities, students extend whole number multiplication and division to ratios and rates. Thus, they expand the repertoire of problems that they can solve by using multiplication and division, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.

Grade 4. Number and Operations and Algebra: Developing quick recall of multiplication facts and related division facts and fluency with whole number multiplication. Students use understandings of multiplication to develop quick recall of the basic multiplication facts and related division facts. They apply their understanding of models for multiplication (i.e., equal-sized groups, arrays, area models, equal intervals on the number line), place value, and properties of operations (in particular, the distributive property) as they develop, discuss, and use efficient, accurate, and generalizable methods to multiply multidigit whole numbers. They select appropriate methods and apply them accurately to estimate products or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for multiplying whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems.

Number and Operations: Developing an understanding of decimals, including the connections between fractions and decimals. Students understand decimal notation as an extension of the base-ten system of writing whole numbers that is useful for representing more numbers, including numbers between 0 and 1, between 1 and 2, and so on. Students relate their understanding of fractions to reading and

writing decimals that are greater than or less than 1, identifying equivalent decimals, comparing and ordering decimals, and estimating decimal or fractional amounts in problem solving. They connect equivalent fractions and decimals by comparing models to symbols and locating equivalent symbols on the number line.

Grade 5. Number and Operations and Algebra: Developing an understanding of and fluency with division of whole numbers. Students apply their understanding of models for division, place value, properties, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multidigit dividends. They select appropriate methods and apply them accurately to estimate quotients or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for dividing whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. They consider the context in which a problem is situated to select the most useful form of the quotient for the solution, and they interpret it appropriately.

Number and Operations: Developing an understanding of and fluency with addition and subtraction of fractions and decimals. Students apply their understandings of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They apply their understandings of decimal models, place value, and properties to add and subtract decimals. They develop fluency with standard procedures for adding and subtracting fractions and decimals. They make reasonable estimates of fraction and decimal sums and differences. Students add and subtract fractions and decimals to solve problems, including problems involving measurement.

Grade 6. Number and Operations: Developing an understanding of and fluency with multiplication and division of fractions and decimals. Students use the meanings of fractions, multiplication and division, and the inverse relationship between multiplication and division to make sense of procedures for multiplying and dividing fractions and explain why they work. They use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain the procedures for multiplying and dividing decimals. Students use common procedures to multiply and divide fractions and decimals efficiently and accurately. They multiply and divide fractions and decimals to solve problems, including multistep problems and problems involving measurement.

Number and Operations: Connecting ratio and rate to multiplication and division. Students use simple reasoning about multiplication and division to solve ratio and rate problems (e.g., "If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12"). By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative sizes of quantities, students extend whole number multiplication and division to ratios and rates. Thus, they expand the repertoire of problems that they can solve by using multiplication and division, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.

III. Measurement

The NCTM standards (2000) define the purpose of the measurement strand as enabling students to "understand measurable attributes of objects and the units, systems, and processes of measurement." Measurement is a common activity that helps students understand the value and uses of mathematics. Finding the length of objects, calculating the area of a room, knowing the volume of a liquid, paying in money, finding birthdays on a calendar, knowing the time, and weighing oneself all involve mathematics topics within the measurement strand.

Measurement inherently involves comparisons. Heavy, heavier, or heaviest or short, shorter, and shortest are relational statements about some measurable attribute of an object. The comparison takes on precision when measurement is expressed in quantifiable standard units. Specifically, measurement involves three steps:

- Determining what attribute to measure
- Selecting the most appropriate attribute unit to measure the attribute
- Applying the unit repeatedly to determine the measure

As students progress through the grades, their breadth and depth of knowledge about measurement increase. Students progress from

- size comparisons to quantitative differences;
- nonstandard units to standard units;
- linear measures to measures of area and volume;
- geometric measurement of length to area to volume to derived measures such as speed;
- standard base units (e.g., meter) to smaller (e.g., millimeter) and larger (e.g. kilometer) units;
- mechanical measurement to measurement as repeated application of standard units; and
- measurement using instruments to measurement using formulas.

Composite Standards

These progressions of learning objectives are incorporated into the topics for measurement as follows: length and distance, perimeter and area, volume, and nongeometric measurement.

Length and distance. As shown in Table 14, linear measurement consists of two categories: the concept of length and tools/measuring length. For the concept of length topic:

- In grade 1, the concepts of the length of an object and the distance between objects are introduced in comparative terms of short, shorter, and shortest or long, longer, and longest.
- The development of units of length begins in grade 1 with centimeter as the unit of length. Successive grades introduce smaller and larger units. In grade 2, students learn about the larger measuring unit of meter, which is 100 centimeters. This concept aligns with introducing 100 in grade 2 in the number strand. The kilometer and millimeter, measures bigger and smaller than a centimeter by factors of 1,000, are introduced in grade 3, which matches students' learning of 1,000.
- In grade 5, students learn conversion from larger to smaller units or vice versa in decimals, which typically is the grade in which students are exposed to decimals.

The tools for measuring and the actual measuring of length topic are continually reinforced throughout the process of learning the concept of length. The applications for length include measuring with different units, estimating lengths, and selecting the appropriate measurement tools (e.g., size of ruler).

Table 14. Composite Standards: Measurement—Length and Distance

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Concept of length	• Understand concepts of length and distance • Understand long, longer, longest, short, shorter, shortest • Understand centimeter	• Understand that a meter is greater than a centimeter	• Understand that a kilometer is greater than a meter and that a millimeter is smaller than a centimeter • Convert compound units to a smaller or a larger unit		• Convert from a smaller unit to a larger unit and vice versa in decimal form	
Tools/measuring length	• Measure and compare lengths of objects and distance with centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in meters and centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in kilometers and millimeters • Estimate lengths and distances • Measure length with appropriate tools			

Perimeter and area. Recognizing that perimeter is a measure of length, Hong Kong, Korea, and Singapore treat perimeter and area together because they both measure different attributes of two-dimensional shapes. The three Asian countries differ slightly in when students first encounter these concepts, with Singapore beginning in grade 3 and Hong Kong and Korea in grade 4. Our measurement composite elects to follow the Singapore approach beginning in grade 3 because this is consistent with the spread of the learning progression across grades. Table 15 shows the composite set of standards for perimeter and area.

- **Perimeter.** The concept of perimeter is introduced in grade 3 for regular and irregular two-dimensional shapes. The initial focus of perimeter calculations is on squares and rectangles in tandem with area. Grade 4 adds complexity to the understanding of perimeter by stressing the relationship between perimeter and area wherein students must find one dimension of a rectangle or a square given information about the area or other dimensions of the perimeter. The concept of circumference (the perimeter of a circle) is addressed in grade 6 in combination with the concept of pi and the calculation of the area of a circle.
- **Area.** The development of the concept of area begins in grade 3 with the introduction of the idea of square units of measurement (square centimeters). By grade 4, the formulas for the areas of rectangles and squares are developed and understanding is deepened by standards that require students to find areas of composite figures made up of rectangles and squares. In grade 5, students are introduced to the area of nonrectangular figures with straight sides, including parallelograms, triangles, and rhombuses. Circles, including finding the area of circles, are introduced in grade 6 after students have been exposed to decimals, which is required for multiplication involving pi.

Table 15. Composite Standards: Measurement—Perimeter and Area

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr.6
Perimeter				• Develop the concept of perimeter • Measure and find the perimeter of 2-dimensional shapes	• Find one dimension of a rectangle given the other dimension and area and perimeter	• Understand the area and the circumference of circles • Understand the concept of pi • Find the area and the circumference of circles, semicircles, and quarter circles
Area			• Develop the concept of area • Compare areas, using improvised units • Measure area in square centimeters (cm^2) and square meters (m^2)	• Apply the formula for area of squares and rectangles and composite figures made up of rectangles and squares	• Apply the formula for the area of triangles, parallelograms, and rhombuses	

Volume. Although the Singapore standards mention volume in the form of capacity in liters in grade 2, the concept of volume is formally emphasized in grade 3 by all three standards. Table 16 shows the composite set of standards for volume.

- **Volume concept.** In grade 3, students are introduced to the concept of capacity, the need for standardized units of capacity (volume), and measurement in liquid units of liter and millimeter. In grade 5, students are introduced to cubic measurement of volume along with the idea that a cubic unit involves the third power, which is not needed for liquid volume units in liters. The cubic centimeter is introduced first and then the cubic meter for measuring larger containers.
- **Volume measurement.** Once again, the actual measuring of capacity (volume) and the use of tools parallel the development of the concept of volume. In grade 3, students encounter only liquid measure and the units of liter and milliliter. In grade 5, students encounter the formulas for finding cubic centimeters. The understanding of volume expands in grade 6 when challenges of finding an unknown volume dimension given volume and information about the length or height dimensions of a container are posed.

Table 16. Composite Standards: Measurement—Volume

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Concept			• Develop the concept of capacity and volume • Understand the need for standardized units of measurement • Understand the units of liter and milliliter		• Introduce the standard unit cubic centimeter (cm^3) • Understand the need for using a unit larger than a cubic centimeter: cubic meter (m^3)	
Tools/measuring			• Measure and compare the volumes of containers using liter and milliliter • Measure volume with appropriate tools		• Measure and compare objects using cm^3 • Understand and apply the formula for finding the volume of cubes and cubes	• Find the length of one edge of a cube given its volume or find the height of a cube given its volume and base area

Nongeometric measurement. The measurement of such nongeometric attributes as time, money, and weight are all familiar attributes, at least in qualitative form, to even young children. A common approach across the standards is to introduce these notions in grade 1 and then develop the concepts and solve more complex problems in later grades. Hence, nongeometric measurement gives students an opportunity to reinforce arithmetic skills. Table 17 shows the composite set of standards for nongeometric measurement.

- Telling and writing clock times to the hour and half hour begin in grade 1. Telling time is refined to 5-minute intervals, and the a.m./p.m. notation is introduced in grade 2. This is followed in grade 4 by telling time to 1-minute intervals, subtracting and adding time in grade 3, measuring time in seconds, and using a 24-hour clock.
- Calendar time begins in grade 1 with the introduction of days of the week and weeks of the year. By grade 2, the standards require students to recognize the number of days in a month and a year, which reinforces students' grade 2 knowledge of numbers up to 1,000.
- Knowing and using money amounts of bills and coins reinforce arithmetic skills in the early grades. The grade 1 standards expose students to cents up to $1 and dollars up to $100. The grade 2 standards introduce students to dollars and cents in decimal notation. Grade 3 introduces problems involving addition and subtraction of money, and grade 4 adds word problems that include multiplication and division of money amounts (e.g., total costs knowing price and quantity).
- Weight is also a familiar attribute that the standards introduce in grade 1 with the concept of weight and expressions in qualitative terms (e.g., heavy, heavier, heaviest). In grade 2, standard units of gram and kilogram for measuring weights are introduced as is the selection of different tools appropriate to the unit (i.e., scales). These understandings are extended in grade 3 through conversion of measurement in compound units between kilograms and grams.

Table 17. Composite Standards: Measurement—Nongeometric

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Time: Clock	• Tell and write time to the hour and half hour	• Tell and write time to 5 minutes • Use a.m. and p.m.	• Tell and write time to 1 minute • Solve word problems involving adding and subtracting time down to the minute	• Measure time in seconds • Use a 24-hour clock to solve clock word problems		
Time: Calendar	• Learn the days of the week • Recognize that there are 12 months in a year	• Recognize the number of days in a month and a year • Understand the relation among 1 hour, 1 day, 1 week, 1 month, and 1 year				
Money	• Tell the amount of money in cents up to $1 and in dollars up to $100 (Excludes combination of dollars and cents)	• Read and write money in decimal notation	• Solve problems involving adding and subtracting money in decimal notation	• Solve word problems involving the four operations and money in decimal notation		
Weight	• Develop the concept of weight • Compare the weights of concrete objects • Express heavy and light and use the terms "heavy," "heavier," and "heaviest"	• Understand the need for using standard units • Measure and compare the weights of objects using gram (g) and kilogram (kg) • Choose the appropriate tools for measuring			• Convert a measurement from a smaller unit to a larger unit in decimal form and vice versa, using kilograms and grams	

NCTM Curriculum Focal Points

NCTM identifies 3 focal points for measurement out of 18 focal points for grades 1–6 (see Table 18). The strength of these 3 measurement focal points is their attention to conceptual understanding of certain measurement ideas and connections with arithmetic. However, the focal points are very limited in delineating the development of topics and learning progressions that

are required for developing a deep understanding of measurement and solving its diverse applications. For example:

- Linear measurement is only alluded to in grade 1. The stress is on conceptual understanding of length by laying multiple copies of a unit end to end. This focal point also emphasizes counting units by using groups of tens, which reinforces arithmetic and place value and an understanding of number lines. However, the discussion does not deal with the fundamental progression in the development and manipulation of different linear measurement units.
- The grade 3 focal point does a good job of stressing an understanding of the concept of area as finding the total number of same-sized units that cover a shape. It also makes a connection to the computation of the area of a rectangle with multiplication. However, omitted is any mention of the computation of area for nonrectangular figures such as triangles. There is no discussion of units of different sizes in relation to the size of an object and only a general sentence about selecting appropriate units and tools for measurement and estimating area—topics with which U.S. students struggle.
- The grade 5 measurement focal point addresses three-dimensional shapes, including volume and surface area. The discussion does a good job of explaining cubic volume. The topics cover tools for solving volume problems, estimating volumes, and decomposing three-dimensional shapes to finding surface areas. However, there is no mention of liquid and cylindrical volume and no systemic build-up of different-sized measurement units.

Table 18. NCTM Focal Points: Measurement

Grade 1. Measurement and Data Analysis: Children strengthen their sense of number by solving problems involving measurements and data. Measuring by laying multiple copies of a unit end to end and then counting the units by using groups of tens and ones support children's understanding of number lines and number relationships. Representing measurements and discrete data in picture and bar graphs involves counting and comparisons that provide another meaningful connection to number relationships.
Grade 3. Measurement: Developing an understanding of area and determining the areas of two-dimensional shapes. Students recognize area as an attribute of two-dimensional regions. They learn that they can quantify area by finding the total number of same-sized units of area that cover the shape without gaps or overlaps. They understand that a square that is 1 unit on a side is the standard unit for measuring area. They select appropriate units, strategies (e.g., decomposing shapes), and tools for solving problems that involve estimating or measuring area. Students connect area measure to the area model that they have used to represent multiplication, and they use this connection to justify the formula for the area of a rectangle.
Grade 5. Geometry and Measurement and Algebra: Describing three-dimensional shapes and analyzing their properties, including volume and surface area. Students relate two-dimensional shapes to three-dimensional shapes and analyze properties of polyhedral solids, describing them by the number of edges, faces, or vertices as well as the types of faces. Students recognize volume as an attribute of three-dimensional space. They understand that they can quantify volume by finding the total number of same-sized units of volume that they need to fill the space without gaps or overlaps. They understand that a cube that is 1 unit on an edge is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating or measuring volume. They decompose three-dimensional shapes and find surface areas and volumes of prisms. As they work with surface area, they find and justify relationships among the formulas for the areas of different polygons. They measure necessary attributes of shapes to use area formulas to solve problems.

IV. Geometry

Geometry is a part of mathematics concerned with questions of size, shape, and relative position of figures and with properties of space. The NCTM standards (2000) identify four geometric abilities that standards should foster:

- Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
- Specify locations and describe spatial relationships using coordinate geometry and other representational systems.
- Apply transformations and use symmetry to analyze mathematical situations.
- Use visualization, spatial reasoning, and geometric modeling to solve problems.

Because geometry is about representing familiar physical objects, it is a natural subject to introduce in the early grades as a bridge to more-abstract mathematical ideas. Because children see the world in 3-D, it also natural to introduce three dimensions early in their schooling.

However, this everyday familiarity with geometric objects can also generate misconceptions that must be addressed and corrected by classroom instruction. For example, students may have a very specific or narrow image in their mind for a geometric concept (e.g., right angle triangle) and not realize the multiple forms it can take on (Vinner & Hershkowitz, 1980). Accordingly, the initial introduction of geometric concepts needs to illustrate two- and three-dimensional shapes with a variety of illustrations of the multiple forms for particular shapes and objects. Students' understanding of the geometry of shapes is honed by being exposed to larger shapes that are composites of simpler shapes and by decomposing larger shapes into smaller shapes and vice versa.

The idea of angles is also essential to initially understanding and solving problems involving geometric shapes, including parallel and intersecting lines, right angle shapes, and circles. The idea of angles is well illustrated as a familiar turning of various amounts.

Composite Standards

The learning progressions that emerge from Hong Kong, Korea, and Singapore work within and across the four core geometric topics of two-dimensional shapes, three-dimensional shapes, lines, and angles.

Two-dimensional shapes. The foundation for developing geometric competencies with two-dimensional shapes begins with a recognition of basic two-dimensional figures. The learning progression for two-dimensional figures then expands with the introduction of properties of angles, parallel and perpendicular lines, and symmetry. Table 19 shows the composite set of standards for two-dimensional shapes.

- The grade 1 standards focus on identifying and naming the four basic two-dimensional shapes: rectangle, square, circle, and triangle. Identification includes finding two-dimensional objects in three-dimensional shapes and completing patterns that vary according to the attributes of shape, size, and color.
- The grade 2 standards extend two-dimensional concepts to circles and semicircles. They also include having students physically copy figures on a dot grid and extend geometric patterns by identifying the orientation of shapes in addition to size and color.
- Building on students' exposure to angles and lines in grade 3, the grade 4 standards focus on properties of right angles, including rectangles and squares. Also introduced in grade 4

is the idea of symmetry of two-dimensional figures, including horizontal and vertical symmetry.

- The progression in grade 5 expands to an understanding of the application of angles in different-shaped triangles and incorporating knowledge of right angles and the sum of angles of a triangle. Also addressed is the application of angles and parallel lines to properties of parallelograms, rhombuses, and trapezoids, including sums of angles and construction.

Table 19. Composite Standards: Geometry—Two-Dimensional Shapes

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Four basic shapes: rectangle, square, circle, triangle • Identify and name the four basic shapes from 2-dimensional and 3-dimensional objects, describing and classifying shapes Patterns: • Make or complete patterns with 2-dimensional cut-outs according to one or two of the following attributes: – *shape* – *size* – *color*	• Identify the basic shapes that make up a given figure • Form different 2-dimensional figures with cut-outs of – *rectangle* – *square* – *triangle* – *semicircle* – *quarter circle* • Copy figures on dot grid or square grid Patterns: • Make or complete patterns with 2-dimensional cut-outs according to one or two of the following attributes: – *shape* – *size* – *orientation* – *color*		Rectangle and square: • Understand the properties of a rectangle and a square • Find unknown angles Symmetry: • Identify symmetric figures • Determine whether a straight line is a line of symmetry of a symmetric figure and complete a symmetric figure with respect to a given horizontal or vertical line of symmetry	Triangle: • Identify and name the following types of triangles: – *isosceles triangle* – *equilateral triangle* – *right angle triangle* • Use the property that the angle sum of a triangle is 180° to find unknown angles • Draw a triangle from given dimensions using ruler, protractor, and set squares Parallelogram, rhombus, and trapezoid: • Identify and name parallelogram, rhombus, and trapezoid • Understand the properties of parallelogram, rhombus, and trapezoid • Find unknown angles • Draw a square, rectangle, parallelogram, rhombus, or trapezoid from given dimensions using ruler, protractor, and set squares	

Three-dimensional shapes. The three-dimensional competencies initially focus on identifying different three-dimensional shapes. After students are exposed to angles (see below), the standards examine the underlying structures of prisms and pyramids in detail. Table 20 shows the composite set of standards for three-dimensional shapes.

- Grade 1 standards introduce three-dimensional shapes of prisms, pyramids, and spheres and employ the skill of completing patterns as a way to reinforce students' understanding and practices in identifying and differentiating among, cubes, cones, and cylinders.
- Grade 2 extends the three-dimensional shapes to include noncubic cylinders. Students learn to recognize faces and construct three-dimensional figures from basic three-dimensional figures such as cubes and cones.
- After introducing angles and two-dimensional figures in grades 3–5, the geometry standards stress three-dimensional figures in grade 6. One set of competencies focuses on identifying the characteristics of polygon-based three-dimensional objects of cubes, prisms, and pyramids; making various three-dimensional polygon shapes from building blocks; and identifying the nets of polygon-based solids. A second set of competencies focuses on understanding the essential characteristics that distinguish cylinders and cones and the concept of a solid of a revolution.

Table 20. Composite Standards: Geometry—Three-Dimensional Shapes

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
• Recognize prisms, pyramids, and spheres • Identity 3-dimensional shapes intuitively • Group 3-dimensional shapes • Describe the relative positions of two 3-dimensional shapes briefly • Make or complete patterns with 3-dimensional models, including cube (rectangular block), cone, and cylinder	• Identify prisms, cylinders, pyramids, and cones • Recognize faces • Group 3-dimensional shapes • Make 3-dimensional shapes • Form different 3-dimensional figures with concrete models of – *cube* – *cone* – *cylinder*				• Understand the concepts of prisms and pyramids and their components and properties • Work with various solid figures: – *Looking at a solid figure made by building blocks, count the number of blocks used* – *Make various shapes using building blocks, and find the patterns* – *Express the shape of a solid figure made by building blocks from the top, front, and side* • Identify nets of solids—cube, prism, and pyramid—and make 3-dimensional solids from given nets • Understand the concepts of cylinders and cones and their components and properties • Understand the concept of a solid of revolution

Lines. Lines are an essential feature for understanding polygon shapes including parallel and perpendicular features of these shapes. Table 21 shows the composite set of standards for lines.

- Lines are introduced in grade 2 and are linked to understanding the shapes of flat faces of three-dimensional figures.
- The concepts of parallel and perpendicular lines are introduced in grade 3 and form the basis for identifying different two-dimensional (e.g., parallelogram) shapes.

Table 21. Composite Standards: Geometry—Lines

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	• Identify lines (straight lines) and curves • Identify edges and faces of a 3-dimensional object	• Identify and draw perpendicular and parallel lines			

Angles. Angles are another essential feature of shapes and underlie the concept of parallel and perpendicular lines. Table 22 shows the composite set of standards for angles.

- Defined as the amount of turning, angles are introduced in grade 3 and identified by their form in different two-dimensional and three-dimensional objects. The notion of right angle is introduced as a formal introduction to perpendicular lines, which are also introduced in grade 3.
- At grade 4, understanding angles expands to understanding the relation between the amount of turn of an angle and degrees, including 90, 180, 270, and 360 degrees, along with knowing the angles of key points on a compass.

Table 22. Composite Standards: Geometry—Angles

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
		• Identify angle as an amount of turning • Identify angles in 2-dimensional and 3-dimensional objects • Identify right angles and angles greater than/smaller than a right angle	• Use notation such as ∠ABC and ∠x to name angles • Estimate and measure angles in degrees • Draw an angle using a protractor • Associate 1/4 turn/right angle with 90°; 1/2 turn with 180°; 3/4 turn with 270°; complete turn with 360°; and 8-point compass		

NCTM Curriculum Focal Points

The strength of the three geometry-related focal points is their stress on the active mathematical process skills—investigating, describing, reasoning, constructing, and deconstructing. The weakness is that the focal points fail to outline the systematic conceptual and procedural development of content across grades required to achieve these understandings (see Table 23).

- Grade 1 stresses composing and decomposing plane and solid figures to recognize perspectives and properties determining how different shapes are alike or different. However, the systemic attention to the definitional development of different shapes over successive grades is omitted.
- Grade 3 geometry emphasizes two-dimensional shapes, again focusing on the physical geometric processes of decomposing, combining, and transforming two-dimensional shapes and applications involving congruence and symmetry. Notably missing is any discussion of angles (except classifying shapes by their sides and angles), and the absence of angles greatly limits two-dimensional geometry in the elementary grades.
- The grade 6 focal point on volume stresses defining the properties of different solids (e.g., number of edges, types of faces), understanding the concept of volume, decomposing three-dimensional shapes, and finding surface area and volume. By integrating geometry

and the measurement of volume and surface area, the focal point does a good job of linking the justifications of the formulas to the shapes of different solids. This is not found in the Asian standards owing to the separation of geometry and measurement.

Table 23. NCTM Focal Points: Geometry

Grade 1. Geometry: Composing and decomposing geometric shapes. Children compose and decompose plane and solid figures (e.g., by putting two congruent isosceles triangles together to make a rhombus), thus building an understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine figures, they recognize them from different perspectives and orientations, describe their geometric attributes and properties, and determine how they are alike and different, in the process developing a background for measurement and initial understandings of such properties as congruence and symmetry.
Grade 3. Geometry: Describing and analyzing properties of two-dimensional shapes. Students describe, analyze, compare, and classify two-dimensional shapes by their sides and angles and connect these attributes to definitions of shapes. Students investigate, describe, and reason about decomposing, combining, and transforming polygons to make other polygons. Through building, drawing, and analyzing two-dimensional shapes, students understand attributes and properties of two-dimensional space and the use of those attributes and properties in solving problems, including applications involving congruence and symmetry.
Grade 5. Geometry and Measurement and Algebra: Describing three-dimensional shapes and analyzing their properties, including volume and surface area. Students relate two-dimensional shapes to three-dimensional shapes and analyze properties of polyhedral solids, describing them by the number of edges, faces, or vertices as well as the types of faces. Students recognize volume as an attribute of three-dimensional space. They understand that they can quantify volume by finding the total number of same-sized units of volume that they need to fill the space without gaps or overlaps. They understand that a cube that is 1 unit on an edge is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating or measuring volume. They decompose three-dimensional shapes and find surface areas and volumes of prisms. As they work with surface area, they find and justify relationships among the formulas for the areas of different polygons. They measure necessary attributes of shapes to use area formulas to solve problems.

V. Data Analysis

The information technology revolution and the avalanche of data we face have transformed the modern-day workplace so that understanding data is an essential skill for a wide variety of jobs. Administrative staff now need to understand how to build spreadsheets. Auto mechanics have to understand how to read car electronic systems and be able to respond appropriately. School teachers and principals have to understand large amounts of student formative and summative data on student performance at the individual and classroom levels.

Composite Standards

The data analysis standards develop two types of student competencies. First is the competency to select and create graphical representations of data. A graph is a visual picture of data that shows patterns of highs and lows and relationships among data items. Students in the elementary grades are exposed to different ways to represent data depending on such data features as whether data are continuous (line chart), in categories (pictograph or bar chart), parts of a whole (circle chart), or grouped (tabular display). Second is the competency to summarize data in terms of averages or measures of central tendency.

The composite data analysis standards address these competencies through exposure to seven topics: classification of objects, pictograms, bar graphs, tables, line graphs, averages, and pie charts. The three sets of Asian country standards follow a somewhat similar progression of topics—pictographs are presented early; bar graphs, tables, and line graphs are presented next; and statistical averages and pie charts appear in the upper primary grades. Another common feature of each of the three sets of data analysis standards is the focus on one or at most two

types of displays at each grade. This fits with the Asian approach to standards in which a topic is taught in depth and then the follow-on standards build on and do not repeat the topic. Also, probability concepts are commonly included in U.S. elementary mathematics standards, but exposure to probability in Asian countries is a topic left for middle school.

Classifying objects. Data must be organized to be understood. In grade 1, the organization of objects into categories and the tallying of objects by category are included as an initial introduction to statistical ideas. For example, when children see a pictures of 6 blue birds and 4 red birds on a page, they may be asked to write how many birds are blue and how many are red and then identify which color group has the largest number and which the smallest number. This categorization is an introduction to simple statistical categorization of data. See Table 24.

Table 24. Composite Standards: Data Analysis—Classifying Objects

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
• Classify objects or people by a predetermined standard, and count the numbers in each category					

Pictograms. Pictograms (grade 2) introduce data representation in a concrete form through pictures. The pictures usually are of the objects the data represent. The standards require students to read and interpret information in different problems by interpreting the scales. For example, in the blue and red bird example above, the pictogram standards start with the actual representation (e.g., 6 blue birds stacked in a column and 4 red birds). The pictures become more abstract when a scale is associate with each picture, such as having each bird in the picture actually represent two birds. See Table 25.

Table 25. Composite Standards: Data Analysis—Pictograms

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	• Compare the quantity of three or more types of objects by arranging them in lines • Read, construct, and interpret picture graphs with scales • Solve problems using information presented in picture graphs				

Bar graphs. Bar graphs (grade 3) employ a series of parallel bars to show the frequency of two or more categories of data. The height of the bar shows the frequency. Asian standards place a major emphasis on bar graphs at grade 3, where bar graphs are introduced as block graphs with 1 square representing 1 unit average value and where students learn that the value of the height of the bar is determined by reading a scale and the scales may be horizontal or vertical. Students develop and interpret their own bar graphs from data. See Table 26.

Table 26. Composite Standards: Data Analysis—Bar Graphs

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
		• Read/discuss block graphs in which 1 square represents 1 unit, average value • Read, construct, and interpret bar graphs in both horizontal and vertical forms, including using their scales • Solve problems using information presented in bar graphs			

Tables. Tables (grade 4) are a powerful way to organize data, especially data involving large quantities of information. Tables are also a first step in organizing information for bar graphs

because they determine data categories. Students must solve problems by reading and interpreting information from tables. See Table 27.

Table 27. Composite Standards: Data Analysis—Tables

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
			• Complete a table from given data • Read and interpret tables • Solve problems using information presented in tables		

Line graphs. Line graphs (grade 4) display continuous data. In elementary school, the displays are typically limited to showing how data change over time, so the line graphs are a series of broken straight lines rather than continuous curves. The standards suggest building on bar graph knowledge by comparing line graphs with bar graphs so that students understand the properties of each and can relate and interpret them to data found in daily life. See Table 28.

Table 28. Composite Standards: Data Analysis—Line Graphs

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
			• Collect data of continuous variates and express them in a graph of broken lines • Compare bar graphs and the graphs of broken lines to understand the properties and uses of each graph		

Averages. One important way that statistics supports an understanding of what data reveal about a phenomenon is by summarizing the data in terms of central tendencies. Averages in Asian standards focus on calculating means—the physical balance point among numbers. Interestingly, other forms of averages, including the median and the mode, are not stressed but are saved for middle school where frequency distributions are presented. Averages are used to reinforce arithmetic and introduce simple algebraic notations such as when the standards require finding the total amount given the average and number of items. See Table 29.

Table 29. Composite Standards: Data Analysis—Averages

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
				• Interpret average as "total amount ÷ number of items" • Calculate the average number/quantity • Solve word problems involving average, including finding the total amount given the average and the number of items	

Pie charts. These circle charts are introduced in grade 6 and correspond with the teaching of circles in geometry. Circle charts visually represent parts of a whole. See Table 30.

Table30. Composite Standards: Data Analysis—Pie Charts

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
					• Read and interpret pie charts • Solve 1-step problems using information presented in pie charts

NCTM Curriculum Focal Points

Although data analysis is an area of relative strength on international assessments for U.S. students, NCTM identifies data analysis only as a connection to the core focal points instead of singling out data analysis as a central element of the core. Thus, the focal points stress data analysis in grade 1 only in terms of strengthening number sense. In grade 3, the focal points link data analysis to using addition, subtraction, multiplication, and division. The grade 4 discussion of data analysis stresses place value, while in grade 5, data analysis is linked to operations with whole numbers, fractions, decimals, and ordered pairs on coordinate grids. These connections are a valuable addition to the detailed Asian data analysis standards but are not a substitute.

More specifically, grade 1 of the focal points discusses representing data in both picture and bar graphs, whereas the Asian standards introduce only picture graphs. Grade 3 discusses frequency tables, bar graphs, picture graphs, and line plots in no particular order and with no development. By grade 5, double-bar graphs are introduced but again with no development, and they are linked with line graphs but without a rationale. See Table 31.

Table 31. NCTM Focal Points: Data Analysis

Grade 1. Measurement and Data Analysis: Children strengthen their sense of number by solving problems involving data. Representing measurements and discrete data in picture and bar graphs involves counting and comparisons that provide another meaningful connection to number relationships.
Grade 3. Addition, subtraction, multiplication, and division of whole numbers come into play as students construct and analyze frequency tables, bar graphs, picture graphs, and line plots and use them to solve problems.
Grade 4. Students continue to use tools from grade 3, solving problems by making frequency tables, bar graphs, picture graphs, and line plots. They apply their understanding of place value to develop and use stem-and-leaf plots.
Grade 5. Students apply their understanding of whole numbers, fractions, and decimals as they construct and analyze double-bar and line graphs and use ordered pairs on coordinate grids.

VI. Algebra

Algebra involves the use of symbols, often in the form of letters, to represent numbers and procedures for working with expressions involving numbers and letters. Unlike NCTM and U.S. state standards, which introduce algebra beginning in grade 1 in the form of pattern completion, Asian mathematics standards introduce algebra in the upper elementary grades and focus on symbolic representations for unknown numbers.

Understanding symbolic representations of numbers can be challenging. Students may interpret $5x$ as $5 + x$ or as 50 plus x, thinking that the 5 is in the tens place and the x is in the ones place. Students may also have difficulty understanding that a letter can represent a set of unknown numbers because they have seen letters as fixed numbers, such as labels on points of a graph. Students may also believe that the expression $4x + 5y$ does not make sense because x and y are different kinds of objects ("and you can't add apples and oranges). In addition, analogies with basic arithmetic rules also may not apply. For example, $20 + 7$ is 27, but $20 + b$ is not $20b$.

Interpreting and solving algebraic expressions introduce unfamiliar ways for students to think about mathematical expressions. In arithmetic, the equal sign is often conceived as "perform a

computation" in the direction of the equal sign, such as 5 + 9 = ___. In algebra, the equal sign means equivalence, and the computation may go in both directions.

Composite Standards

These examples demonstrate the importance of algebraic standards that stress an understanding of symbolic representation and the reasons underlying the procedures for solving algebraic equations. See Table 32.

Expressions. The use of symbols and expressions to represent numbers is introduced in grade 5. A variety of symbolic expressions are presented to expose students to different uses of symbolic representations. In the example, the *x* represents some number that is not yet known. That is, whatever *x* is currently, add 10 to find out the person's age 10 years from now.

Equations. The concept of equations as statement that two mathematical expressions is also introduced in grade 5, along with expressions. Grade 5 introduces 1-step problems (e.g., the price of an evening movie is $2 more than the price of an afternoon movie. If the price of an afternoon movie is $6, how much is the price of the evening movie?). Grade 6 introduces 2-step equation problems are (e.g., add to the previous example: If the cashier is handed $20 to pay for the evening movie, how much change is received?).

Table 32. Composite Standards: Algebra

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Expressions					Expressions: • Use symbols or letters to represent numbers • Record with algebraic symbols, for example, "John is *x* years old now. How old will he be after 10 years?" and record as (*x* + 10) years old	
Equations					Simple equations involving 1 step in finding the solution: • Understand the concept of equations • Solve simple equations involving 1 step in the solutions and check the answers (involving whole numbers only) • Solve problems by simple equations (involving only 1 step in the solutions)	Simple equations (involving 2 steps in finding the solution): • Solve equations involving at most 2 steps in the solutions, and examine the results • Solve problems by using simple equations (involving at most 2 steps in the solution)

NCTM Curriculum Focal Points

The focal points' attention to algebra is rather broad compared with the narrow concept of representation of unknown numbers by letters found in the Asian standards. For example, the focal points include algebra as part of numbers in many grades. The focal points also have a section called connections to the focal points that in the early grades stresses understanding patterns.

Grade 1 connections to focal points—Algebra: Through identifying, describing, and applying number patterns and properties in developing strategies for basic facts, children learn about other properties of numbers and operations, such as odd and even (e.g., "Even numbers of objects can be paired, with none left over"), and 0 as the identity element for addition.

The only grades 1–6 focal point expressly focused on algebra is similar to the Asian interpretation of algebra as representing numbers with letters and solving simple equations. In fact, the algebraic focal point may more clearly express the reasons for using letter representations than the Asian standards. The language stresses that "expressions in different forms can be equivalent" and that the different forms may have advantages in "compactness" or "feature different information." This focal point involves only 1 step, not the 2-step problems found in the composite standards. See Table 33.

Table 33. NCTM Focal Points: Algebra

Grade 6. Writing, interpreting, and using mathematical expressions and equations. Students write mathematical expressions and equations that correspond to given situations, they evaluate expressions, and they use expressions and formulas to solve problems. They understand that variables represent numbers whose exact values are not yet specified, and they use variables appropriately. Students understand that expressions in different forms can be equivalent, and they can rewrite an expression to represent a quantity in a different way (e.g., to make it more compact or to feature different information). Students know that the solutions of an equation are the values of the variables that make the equation true. They solve simple one-step equations by using number sense, properties of operations, and the idea of maintaining equality on both sides of an equation. They construct and analyze tables (e.g., to show quantities that are in equivalent ratios), and they use equations to describe simple relationships (such as 3x = y) shown in a table.

VII. Implications for U.S. Standards Development

The composite grades 1–6 mathematics standards based on the standards of high-performing Hong Kong, Korea, and Singapore address the call by the Common Core State Standards Initiative and the federal RttT to develop common standards that are internationally benchmarked. International benchmarking highlights an essential feature of the composite Asian standards: They tend to limit topic repetition and specify mathematical competencies precisely as mathematical content progresses systematically in ways that follow the natural logic of mathematics over grades.

The composite standards developed in this study highlight this progression of mathematical content across grades. This organization departs from the traditional grade-by-grade organization that displays the mathematical content to be taught at each grade. The grade-level organization makes sense when showing what mathematics topics to cover at each grade across mathematical strands. But the organization of standards by grade level fails to illuminate how teachers of mathematical content build on prior topic coverage and progress across grades. To facilitate an understanding of the sequencing of mathematical content, the composite standards are organized by topic and display the progression of mathematical content across grades for each topic.

Overall, the composite standards of the high-performing Asian countries provide a treatment of grade 1–6 mathematics with four key features.

First, the composite standards *concentrate the early learning of mathematics on the numbers, measurement, and geometry strands* with less emphasis on data analysis and little exposure to

algebra. The Hong Kong standards for grades 1–3 devote approximately half the targeted time to numbers and almost all the time remaining to geometry and measurement.

Second, the composite standards *sequence topics* within strands to support in-depth and efficient development of mathematics content following a logical development of mathematical knowledge. For example, the numbers strand sequence progression is whole numbers, arithmetic operations, fractions, decimals, ratios, and percents. Measurement introduces linear measurement followed by perimeter and area (two-dimensional measurement) and then the more-complicated volume (three-dimensional measurement). Geometry initially introduces the features of shapes, proceeds to cover two-dimensional geometry along with angles and parallel lines, and concludes with the features of three-dimensional figures. Data analysis starts with pictograms, a visual and more-familiar way to examine data, and then moves on to bar charts and more-complicated continuous line charts.

Third, the composite standards *sequence mathematical competencies within a topic* across the grades according to a mathematically logical progression. Several illustrations occur within the numbers strand. Whole numbers are ordered by size, with grade 1 addressing numbers up to 100, grade 2 up to 1,000, grade 3 up to 10,000, and grade 4 up to 100,000. Grade 5 emphasizes an understanding of large numbers in general. Multiplication is also carefully developed, with grade 2 starting with the basic multiplication concept and multiplication tables for 2, 3, 4, 5, and 10; grade 3 extends to tables 6, 7, 8, and 9 along with multiplication of one digit by two and three digits; grade 4 introduces associative and commutative properties and multiplication of two-digit numbers by three-digit numbers; and grade 5 covers common multiples and the relation with common divisors.

Fourth, the *ordering of content for one topic is frequently aligned to reinforce the content of another topic* for the same or prior grades. Linear measurement in grade 1 introduces the centimeter, which is aligned with grade 1 exposure to whole numbers up to 100. Grade 3 introduces kilometers and millimeters after 1,000 is taught within the whole-numbers strand of grade 2. Grade 3 introduces the multiplication and division of money (e.g., relations between total costs with price and quantity), thus reinforcing the learning of multiplication and division in grades 2 and 3. Still another example of cross-topic reinforcement occurs within grade 6 data analysis, which introduces pie charts around the same time that circles are introduced in geometry.

In addition, it is important to note that in many cases, particularly within the number strand, the composite standards show a grade placement of a particular skill or concept that is one year earlier than is common in much of the United States. Although this finding is notable, we believe that it is the coherent learning progressions and content connections that are much more important to emulate than the grade placement of particular topics. Further, the delineation of content by learning progressions facilitates an adjustment of the grade placement of content to fit the learning pace of individual students within a common standards framework that all students are eventually expected to master.

The NCTM focal points were also explored by strand because they are designed to provide benchmarks to guide the development of focused and coherent U.S. state standards. The focal points are limited to three benchmarks per grade. The focal points, like the composite standards,

emphasize numbers and cover all major number topics. A particular strength of the focal points is the emphasis on the conceptual development of mathematical concepts. For example, the grade 3 multiplication and division focal points include "students understand the meanings of multiplication and division of whole numbers through the use of representations (e.g., equal-sized groups, arrays, area models, and equal "jumps" on number lines for multiplication, and successive subtraction, partitioning, and sharing for division)."

Although the focal points provide another useful check on the development of common U.S. mathematics standards, a major limitation in using them is that topic content tends to be consolidated at one grade and as such may omit topics or fail to build systematic learning progressions across grades. Thus, the focal point presentation of whole numbers stops with 1,000 and does not systematically introduce 10,000, 100,000, and the understanding of larger numbers in general.

Two limitations of the composite standards should be noted. First, the composite standards are intended to highlight the progression of core mathematical content found in the standards of the three high-performing Asian countries and do not include all the details. For example, calculators, a controversial issue in the United States, do not seem to be an essential feature of the Asian standards, although all three standards introduce calculators in grade 4. For instance, Hong Kong has a grade 4 topic on "acquaintance with modern calculating devices with three competencies: 1. Be familiar with modern calculating devices; 2. Recognize the basic operations and functions of a calculator; and 3. Use calculators to carry out activities to foster pupils' number sense." The only mention of calculators in the Korean standards is at grade 4: "Using a calculator or actual calculation, confirm the estimation." Thus, calculators are not highlighted in the composite standards.

Second, in their 2009 *Education Week* commentary "Standards Are Not Enough," Fuhrman and colleagues point out that with respect to federal aid, standards are only the starting point for instructional reform:

> The new federal money to states, including $350 million set aside specifically for the development of assessments tied to the new standards, will be used best if it helps states create the whole package of tools, all tied to the new standards: curricula, teacher training, instructional materials, and good assessments that give real-time insights into what students are understanding and where they are still struggling, while also providing a final reading on what they have learned.

Their observations suggest that follow-up studies should also look to the international high performers with respect to such matters as teacher preparation for the new curricula, instructional materials, and assessments.

In conclusion, the development of standards is only the front end of a long-term reform process. It is critical that sound standards be written to guide the rest of the reform process, however, we do not imply that merely replicating these composite standards is sufficient. Rather we have presented a set of composite mathematics standards of the three Asian high performers to offer a theoretically and empirically valid international benchmark for the development of common U.S. standards in mathematics. We have explained how these composite standards differ from the

typical curriculum found in the United States and noted how they address some of the recognized deficiencies found in the U.S. grades 1-6 mathematics curricula.

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Appendix

Table A1. Numbers and Operations

Composite Standards: Numbers and Operations						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
Whole Numbers/ Place Value	Whole numbers to 100: • Count to tell the number of objects in a given set • Count forward and backward • Compare the number of objects in two or more sets • Use ordinal numbers (first, second, up to tenth) and symbols (1st, 2nd, 3rd, etc.) • Use number notation and place values (tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers	Whole numbers to 1,000: • Count in tens and hundreds • Use number notation and place values (hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers	Whole numbers to 10,000: • Use number notation and place values (thousands, hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers • Understand odd and even numbers	Whole numbers to 100,000: • Use number notation and place values (ten thousands, thousands, hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers • Round numbers to the nearest 10 or 100	• Develop an understanding of large numbers • Develop the concept of approximation • Estimate the number of a large quantity of objects • Round large numbers in thousands, ten thousands, hundred thousands, millions, ten millions, hundred millions	
Addition/Subtraction	Addition and subtraction: • Understand situations for, and the meaning of,	Addition and subtraction of numbers up to three digits: • Solve up to 2-step	Addition and subtraction of numbers up to four digits: • Use the terms			

Composite Standards: Numbers and Operations						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
	addition and subtraction • Use the addition symbol (+) or the subtraction symbol (−) • Compare two numbers within 20 to tell how much one number is greater (or smaller) than the other • Recognize the relationship between addition and subtraction • Build the addition bonds up to 9 + 9 • Solve 1-step word problems involving addition and subtraction within 20 • Add more than two 1-digit numbers • Add and subtract within 100 without regrouping involving – *a 2-digit number and ones* – *a 2-digit number and*	word problems involving addition and subtraction • Use mental calculation for addition and subtraction involving – *a 3-digit number and ones* – *a 3-digit number and tens* – *a 3-digit number and hundreds*	"sum" and "difference" • Solve up to 2-step word problems involving addition and subtraction			

Composite Standards: Numbers and Operations						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
	tens – *two 2-digit numbers* • Use mental calculation for addition and subtraction – *within 20* – *involving a 2-digit number and ones without renaming* – *involving a 2-digit number and tens*					
Multiplication/Division		Basic multiplication (basic concept and computation): • Understand the situations for, and meaning of, multiplication • Build up the multiplication tables of 2, 3, 4, 5, and 10 • Discover the commutative property of multiplication through concrete examples (e.g., 2 × 3＝3 × 2) Basic division (basic	Multiplication: • Build up the multiplication tables of 6, 7, 8, and 9 • Perform multiplication with a multiplier of 1 digit and a multiplicand of 2 or 3 digits Division: • Understand the situations for, and meaning of, division • Perform basic division by short division	Multiplication: • Discover the associative property of multiplication through concrete examples • Apply the commutative and associative properties of multiplication in computation (e.g., 2 × 8 × 5 = (2 × 5) × 8) • Perform multiplication with a multiplier of 2 digits and a multiplicand of 2	Divisors and multiples: • Understand the meaning of "divisor," "common divisor," and "greatest common divisor" and know how to solve for them • Understand the meaning of "multiple," "common multiple," and "least common multiple" and know how to solve for them • Understand the	

Composite Standards: Numbers and Operations						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
		concept and computation): • Develop the concept of division: sharing and grouping • Divide a quantity (not greater than 20) into equal sets given – *the number of objects in each set* – *the number of sets* • Recognize the relationship between multiplication and division • Solve 1-step word problems involving multiplication and division within the multiplication tables	• Perform division with a divisor of 1 digit and a dividend of 2 or 3 digits with and without remainders • Use the terms "product," "quotient," and "remainder" • Solve up to 2-step word problems involving the four operations, including estimating answers	digits and then 3 digits Division: • Perform division with a divisor of 2 digits and a dividend of 2 and then 3 digits • Recognize divisibility when the divisors are 2, 5, and 10 • Identify 1-digit factors of 2-digit numbers • Distinguish between factors and multiples • Solve up to 3-step word problems involving the four operations, including estimating answers	relation between divisors and multiples and know how to apply them • Multiply and divide by 10, 100, and 1000 mentally • Use order of operations, combined operations involving the four operations, and brackets • Solve word problems involving the four operations, including estimating answers	
Fractions / Concepts		Fraction of a whole: • Interpret a fraction as part of a whole • Read and write fractions • Compare and order unit fractions and like fractions (denominators less than or equal	Equivalent fractions: • Recognize and name equivalent fractions • Write the equivalent fraction of a fraction, given the denominator or the numerator • Express a fraction	Mixed numbers and improper fractions: • Understand the concepts of mixed numbers and improper fractions • Express an improper fraction as a mixed number, and vice		

Composite Standards: Numbers and Operations						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
Fractions/ Arithmetic Operations		to 12)	in its simplest form • Compare and order unlike fractions, including comparing fractions with respect to one half (denominators less than or equal to 12)	versa, and express both in simplest form		
			Addition and subtraction of two related fractions (one denominator a factor of the other) within one whole (denominators of given fractions should not exceed 12)	Addition and subtraction: • Add and subtract like fractions • Add and subtract related fractions (denominators of given fractions should not exceed 12) Multiplication of a proper or improper fraction and a whole number	Addition and subtraction of fractions with unlike denominators: • Add and subtract fractions with unlike denominators Multiplication and division of fractions: • Multiply proper fractions, improper fractions, mixed numbers and whole numbers by proper fractions, improper fractions, and mixed numbers • Divide fractions by whole numbers and whole numbers by fractions	Division of fractions: • Divide proper fractions by proper fractions Mixed calculations with fraction and decimal: • Know how to solve simple calculations with both fractions and decimals

Composite Standards: Numbers and Operations						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
Decimals				Decimals up to three decimal places: • Understand notation and place values (tenths, hundredths, thousandths), including identifying the values of the digits in a decimal • Use the number line to display decimals • Compare and order decimals • Convert a decimal to a fraction • Convert a fraction whose denominator is a factor of 10 or 100 to a decimal • Round decimals to the nearest whole number	Decimal addition and subtraction: • Add and subtract decimals up to two places of decimals and for sums involving at most three operations • Estimate the answers Decimal multiplication: • Develop an understanding of multiplication of decimals through daily life examples • Multiply decimals by whole numbers and by decimals • Estimate the answers	Decimal division: • Develop an understanding of division of decimals through daily life examples • Divide decimals by whole numbers, whole numbers by decimals, and decimals by decimals • Perform mixed operations on decimals for sums involving at most three operations • Estimate the answers Decimal conversion: • Convert decimals to fractions and fractions to decimals • Compare fractions by converting them to decimals • Estimate the answers
Ratio					Ratio (excludes ratios involving fractions and decimals): • Interpret *a* : *b* and	

Composite Standards: Numbers and Operations						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
					a : *b* : *c*, where *a*, *b*, and *c* are whole numbers • Express a ratio in its simplest form • Find the ratio of two or three given quantities • Write equivalent ratios and find the missing term in a pair of equivalent ratios • Solve up to 2-step word problems involving ratio, including finding one quantity given the other quantity and their ratio	
Percents						Percents (basic concept; conversion of percentages to decimals or fractions and vice versa): • Recognize percentages through daily life examples • Develop an understanding of percentages • Convert percentages to decimals and vice versa

Composite Standards: Numbers and Operations						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
						Applications of percents: • Solve simple problems on percentages, including finding percentages expressing the value of a percentage of a quantity applying discounts • Estimate the answers

NCTM Curriculum Focal Points: Numbers and Operations

Grade 1. Number and Operations: Developing an understanding of whole number relationships including grouping in tens and ones. Children compare and order whole numbers (at least to 100) to develop an understanding of and solve problems involving the relative sizes of these numbers. They think of whole numbers between 10 and 100 in terms of groups of tens and ones (especially recognizing the numbers 11 to 19 as 1 group of ten and particular numbers of ones). They understand the sequential order of the counting numbers and their relative magnitudes and represent numbers on a number line.

Number and Operations and Algebra: Developing understandings of addition and subtraction and strategies for basic addition facts and related subtraction facts. Children develop strategies for adding and subtracting whole numbers on the basis of their earlier work with small numbers. They use a variety of models, including discrete objects, length-based models (e.g., lengths of connecting cubes), and number lines, to model "part-whole," "adding to," "taking away from," and "comparing" situations to develop an understanding of the meanings of addition and subtraction and strategies to solve such arithmetic problems. Children understand the connections between counting and the operations of addition and subtraction (e.g., adding two is the same as "counting on" two). They use properties of addition (commutativity and associativity) to add whole numbers, and they create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems involving basic facts. By comparing a variety of solution strategies, children relate addition and subtraction as inverse operations.

Grade 2. Number and Operations: Developing an understanding of the base-ten numeration system and place-value concepts. Children develop an understanding of the base-ten numeration system and place-value concepts (at least to 1000). Their understanding of base-ten numeration includes ideas of counting in units and multiples of hundreds, tens, and ones, as well as a grasp of number relationships, which they demonstrate in a variety of ways, including comparing and ordering numbers. They understand multidigit numbers in terms of place value, recognizing that place-value notation is a shorthand for the sums of multiples of powers of 10 (e.g., 853 as 8 hundreds + 5 tens + 3 ones).

Number and Operations and Algebra: Developing quick recall of addition facts and related subtraction facts and fluency with multidigit addition and

subtraction. Children use their understanding of addition to develop quick recall of basic addition facts and related subtraction facts. They solve arithmetic problems by applying their understanding of models of addition and subtraction (such as combining or separating sets or using number lines), relationships and properties of number (such as place value), and properties of addition (commutativity and associativity).Children develop, discuss, and use efficient, accurate, and generalizable methods to add and subtract multidigit whole numbers. They select and apply appropriate methods to estimate sums and differences or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems.

Grade 3. Number and Operations and Algebra: Developing understandings of multiplication and division and strategies for basic multiplication facts and related division facts. Students understand the meanings of multiplication and division of whole numbers through the use of representations (e.g., equal-sized groups, arrays, area models, and equal "jumps" on number lines for multiplication, and successive subtraction, partitioning, and sharing for division). They use properties of addition and multiplication (e.g., commutativity, associativity, and the distributive property) to multiply whole numbers and apply increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving basic facts. By comparing a variety of solution strategies, students relate multiplication and division as inverse operations.

Number and Operations: Connecting ratio and rate to multiplication and division Students use simple reasoning about multiplication and division to solve ratio and rate problems (e.g., "If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12"). By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative sizes of quantities, students extend whole number multiplication and division to ratios and rates. Thus, they expand the repertoire of problems that they can solve by using multiplication and division, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.

Grade 4. Number and Operations and Algebra: Developing quick recall of multiplication facts and related division facts and fluency with whole number multiplication. Students use understandings of multiplication to develop quick recall of the basic multiplication facts and related division facts. They apply their understanding of models for multiplication (i.e., equal-sized groups, arrays, area models, equal intervals on the number line), place value, and properties of operations (in particular, the distributive property) as they develop, discuss, and use efficient, accurate, and generalizable methods to multiply multidigit whole numbers. They select appropriate methods and apply them accurately to estimate products or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for multiplying whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems.

Number and Operations: Developing an understanding of decimals, including the connections between fractions and decimals. Students understand decimal notation as an extension of the base-ten system of writing whole numbers that is useful for representing more numbers, including numbers between 0 and 1, between 1 and 2, and so on. Students relate their understanding of fractions to reading and writing decimals that are greater than or less than 1, identifying equivalent decimals, comparing and ordering decimals, and estimating decimal or fractional amounts in problem solving. They connect equivalent fractions and decimals by comparing models to symbols and locating equivalent symbols on the number line.

Grade 5. Number and Operations and Algebra: Developing an understanding of and fluency with division of whole numbers. Students apply their understanding of models for division, place value, properties, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multidigit dividends. They select appropriate methods and apply them accurately to estimate quotients or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for dividing whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. They consider the context in which a problem is situated to

select the most useful form of the quotient for the solution, and they interpret it appropriately.

Number and Operations: Developing an understanding of and fluency with addition and subtraction of fractions and decimals. Students apply their understandings of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They apply their understandings of decimal models, place value, and properties to add and subtract decimals. They develop fluency with standard procedures for adding and subtracting fractions and decimals. They make reasonable estimates of fraction and decimal sums and differences. Students add and subtract fractions and decimals to solve problems, including problems involving measurement.

Grade 6. Number and Operations: Developing an understanding of and fluency with multiplication and division of fractions and decimals. Students use the meanings of fractions, multiplication and division, and the inverse relationship between multiplication and division to make sense of procedures for multiplying and dividing fractions and explain why they work. They use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain the procedures for multiplying and dividing decimals. Students use common procedures to multiply and divide fractions and decimals efficiently and accurately. They multiply and divide fractions and decimals to solve problems, including multistep problems and problems involving measurement.

Number and Operations: Connecting ratio and rate to multiplication and division. Students use simple reasoning about multiplication and division to solve ratio and rate problems (e.g., "If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12"). By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative sizes of quantities, students extend whole number multiplication and division to ratios and rates. Thus, they expand the repertoire of problems that they can solve by using multiplication and division, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.

Table A2. Measurement

Composite Standards: Measurement						
Topics	**Gr. 1**	**Gr. 2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr.6**
Linear Measurement						
Concept of length	• Understand concepts of length and distance • Understand long, longer, longest, short, shorter, shortest • Understand centimeter	• Understand that a meter is greater than a centimeter	• Understand that a kilometer is greater than a meter and that a millimeter is smaller than a centimeter • Convert compound units to a smaller or a larger unit		• Convert from a smaller unit to a larger unit and vice versa in decimal form	
Tools/measuring length	• Measure and compare lengths of objects and distance with centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in meters and centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in kilometers and millimeters • Estimate lengths and distances • Measure length with appropriate tools			
Perimeter/Area						
Perimeter				• Develop the concept of perimeter • Measure and find the perimeter of 2-dimensional shapes	• Find one dimension of a rectangle given the other dimension and area and perimeter	• Understand the area and the circumference of circles • Understand the concept of pi • Find the area and the circumference of circles, semicircles, and quarter circles
Area			• Develop the concept of area • Compare areas, using improvised units	• Apply the formula for area of squares and rectangles and composite figures made up of	• Apply the formula for the area of triangles, parallelograms, and rhombuses	

Composite Standards: Measurement						
Topics	**Gr. 1**	**Gr. 2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr.6**
			• Measure area in square centimeters (cm^2) and square meters (m^2)	rectangles and squares		
Volume						
Concept			• Develop the concept of capacity and volume • Understand the need for standardized units of measurement • Understand the units of liter and milliliter		• Introduce the standard unit cubic centimeter (cm^3) • Understand the need for using a unit larger than a cubic centimeter: cubic meter (m^3)	
Tools / Measuring			• Measure and compare the volumes of containers using liter and milliliter • Measure volume with appropriate tools		• Measure and compare objects using cm^3 • Understand and apply the formula for finding the volume of cubes and cubes	• Find the length of one edge of a cube given its volume or find the height of a cube given its volume and base area
Nongeometric Measurement						
Time: Clock	• Tell and write time to the hour and half hour	• Tell and write time to 5 minutes • Use a.m. and p.m.	• Tell and write time to 1 minute • Solve word problems involving adding and subtracting time down to the minute	• Measure time in seconds • Use a 24-hour clock to solve clock word problems		
Time: Calendar	• Learn the days of the week • Recognize that	• Recognize the number of days in a month and a year				

Composite Standards: Measurement						
Topics	**Gr. 1**	**Gr. 2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr.6**
	there are 12 months in a year	• Understand the relation among 1 hour, 1 day, 1 week, 1 month, and 1 year				
Money	• Tell the amount of money in cents up to $1 and in dollars up to $100 (excludes combination of dollars and cents)	• Read and write money in decimal notation	• Solve problems involving adding and subtracting money in decimal notation	• Solve word problems involving the four operations and money in decimal notation		
Weight	• Develop the concept of weight • Compare the weights of concrete objects • Express heavy and light and use the terms "heavy," "heavier," and "heaviest"	• Understand the need for using standard units • Measure and compare the weights of objects using gram (g) and kilogram (kg) • Choose the appropriate tools for measuring			• Convert a measurement from a smaller unit to a larger unit in decimal form and vice versa, using kilograms and grams	

NCTM Curriculum Focal Points: Measurement

Grade 1. Measurement and Data Analysis. Children strengthen their sense of number by solving problems involving measurements and data. Measuring by laying multiple copies of a unit end to end and then counting the units by using groups of tens and ones supports children's understanding of number lines and number relationships. Representing measurements and discrete data in picture and bar graphs involves counting and comparisons that provide another meaningful connection to number relationships.

Grade 3. Measurement: Developing an understanding of area and determining the areas of two-dimensional shapes. Students recognize area as an attribute of two-dimensional regions. They learn that they can quantify area by finding the total number of same-sized units of area that cover the shape without gaps or overlaps. They understand that a square that is 1 unit on a side is the standard unit for measuring area. They select appropriate units, strategies (e.g., decomposing shapes), and tools for solving problems that involve estimating or measuring area. Students connect area measure to the area model that they have used to represent multiplication, and they use this connection to justify the formula for the area of a rectangle.

Grade 5. Geometry and Measurement and Algebra: Describing three-dimensional shapes and analyzing their properties, including volume and surface area. Students relate two-dimensional shapes to three-dimensional shapes and analyze properties of polyhedral solids, describing them by the number of edges, faces, or vertices as well as the types of faces. Students recognize volume as an attribute of three-dimensional space. They understand that they can quantify volume by finding the total number of same-sized units of volume that they need to fill the space without gaps or overlaps. They understand that a cube that is 1 unit on an edge is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating or measuring volume. They decompose three-dimensional shapes and find surface areas and volumes of prisms. As they work with surface area, they find and justify relationships among the formulas for the areas of different polygons. They measure necessary attributes of shapes to use area formulas to solve problems.

Table A3. Geometry

Composite Standards: Geometry						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
2D- Shapes	Four basic shapes: rectangle, square, circle, triangle • Identify and name the four basic shapes from 2-dimensional and 3-dimensional objects, describing and classifying shapes Patterns: • Make or complete patterns with 2-dimensional cut-outs according to one or two of the following attributes: – *shape* – *size* – *color*	• Identify the basic shapes that make up a given figure • Form different 2-dimensional figures with cut-outs of – *rectangle* – *square* – *triangle* – *semicircle* – *quarter circle* • Copy figures on dot grid or square grid Patterns: • Make or complete patterns with 2-dimensional cut-outs according to one or two of the following attributes: – *shape* – *size* – *orientation* – *color*		Rectangle and square: • Understand the properties of a rectangle and a square • Find unknown angles Symmetry: • Identify symmetric figures • Determine whether a straight line is a line of symmetry of a symmetric figure and complete a symmetric figure with respect to a given horizontal or vertical line of symmetry	Triangle: • Identify and name the following types of triangles: – *isosceles triangle* – *equilateral triangle* – *right angle triangle* • Use the property that the angle sum of a triangle is 180^{o} to find unknown angles • Draw a triangle from given dimensions using ruler, protractor, and set squares Parallelogram, rhombus, and trapezoid: • Identify and name parallelogram, rhombus, and trapezoid • Understand the properties of parallelogram, rhombus, and trapezoid • Find unknown angles	

Composite Standards: Geometry						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
					• Draw a square, rectangle, parallelogram, rhombus, or trapezoid from given dimensions using ruler, protractor, and set squares	
3-D Shapes	• Recognize prisms, pyramids, and spheres • Identity 3-dimensional shapes intuitively • Group 3-dimensional shapes • Describe the relative positions of two 3-dimensional shapes briefly • Make or complete patterns with 3-dimensional models, including cube (rectangular block), cone, and cylinder	• Identify prisms, cylinders, pyramids, and cones • Recognize faces • Group 3-dimensional shapes • Make 3-dimensional shapes • Form different 3-dimensional figures with concrete models of – *cube* – *cone* – *cylinder*				• Understand the concepts of prisms and pyramids and their components and properties • Work with various solid figures: – *Looking at a solid figure made by building blocks, count the number of blocks used* – *Make various shapes using building blocks, and find the patterns* – *Express the shape of a solid figure made by building blocks from the top, front, and side*

Composite Standards: Geometry						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
						• Identify nets of solids—cube, prism, and pyramid—and make 3-dimensional solids from given nets • Understand the concepts of cylinders and cones and their components and properties • Understand the concept of a solid of revolution
Lines		• Identify lines (straight lines) and curves • Identify edges and faces of a 3-dimensional object	• Identify and draw perpendicular and parallel lines			
Angles			• Identify angle as an amount of turning • Identify angles in 2-dimensional and 3-dimensional objects • Identify right angles and angles greater than/ smaller than a right angle	• Use notation such as ∠ABC and ∠x to name angles • Estimate and measure angles in degrees • Draw an angle using a protractor • Associate 1/4 turn/right angle with 90°; 1/2 turn with 180°; 3/4 turn with 270°;		

Composite Standards: Geometry						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
				complete turn with 360°; and 8-point compass		

NCTM Curriculum Focal Points: Geometry

Grade 1. Geometry: Composing and decomposing geometric shapes. Children compose and decompose plane and solid figures (e.g., by putting two congruent isosceles triangles together to make a rhombus), thus building an understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine figures, they recognize them from different perspectives and orientations, describe their geometric attributes and properties, and determine how they are alike and different, in the process developing a background for measurement and initial understandings of such properties as congruence and symmetry.

Grade 3. Geometry: Describing and analyzing properties of two-dimensional shapes. Students describe, analyze, compare, and classify two-dimensional shapes by their sides and angles and connect these attributes to definitions of shapes. Students investigate, describe, and reason about decomposing, combining, and transforming polygons to make other polygons. Through building, drawing, and analyzing two-dimensional shapes, students understand attributes and properties of two-dimensional space and the use of those attributes and properties in solving problems, including applications involving congruence and symmetry.

Grade 5. Geometry and Measurement and Algebra: Describing three-dimensional shapes and analyzing their properties, including volume and surface area. Students relate two-dimensional shapes to three-dimensional shapes and analyze properties of polyhedral solids, describing them by the number of edges, faces, or vertices as well as the types of faces. Students recognize volume as an attribute of three-dimensional space. They understand that they can quantify volume by finding the total number of same-sized units of volume that they need to fill the space without gaps or overlaps. They understand that a cube that is 1 unit on an edge is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating or measuring volume. They decompose three-dimensional shapes and find surface areas and volumes of prisms. As they work with surface area, they find and justify relationships among the formulas for the areas of different polygons. They measure necessary attributes of shapes to use area formulas to solve problems.

Table A4. Data Analysis

Composite Standards: Data Analysis						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
Classifying Objects	• Classify objects or people by a predetermined standard, and count the numbers in each category					
Pictograms		• Compare the quantity of three or more types of objects by arranging them in lines • Read, construct, and interpret picture graphs with scales • Solve problems using information presented in picture graphs				

Composite Standards: Data Analysis						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
Bar Graphs			• Read/discuss block graphs in which 1 square represents 1 unit, average value • Read, construct, and interpret bar graphs in both horizontal and vertical forms, including using their scales • Solve problems using information presented in bar graphs			
Tables				• Complete a table from given data • Read and interpret tables • Solve problems using information presented in tables		
Line Graphs				• Collect data of continuous variates and express them in a graph of broken lines • Compare bar graphs and the graphs of broken lines to understand the properties and uses of each graph		

Composite Standards: Data Analysis						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
Averages					• Interpret average as "total amount ÷ number of items" • Calculate the average number/quantity • Solve word problems involving average, including finding the total amount given the average and the number of items	
Pie Charts						• Read and interpret pie charts • Solve 1-step problems using information presented in pie charts

NCTM Curriculum Focal Points: Data Analysis

Grade 1. Measurement and Data Analysis: Children strengthen their sense of number by solving problems involving data. Representing measurements and discrete data in picture and bar graphs involves counting and comparisons that provide another meaningful connection to number relationships.

Grade 3. Addition, subtraction, multiplication, and division of whole numbers come into play as students construct and analyze frequency tables, bar graphs, picture graphs, and line plots and use them to solve problems.

Grade 4. Students continue to use tools from grade 3, solving problems by making frequency tables, bar graphs, picture graphs, and line plots. They apply their understanding of place value to develop and use stem-and-leaf plots.

Grade 5. Students apply their understanding of whole numbers, fractions, and decimals as they construct and analyze double-bar and line graphs and use ordered pairs on coordinate grids.

Table A5. Algebra

Composite Standards: Algebra						
Topics	**Gr. 1**	**Gr.2**	**Gr. 3**	**Gr. 4**	**Gr. 5**	**Gr. 6**
Expressions					Expressions: • Use symbols or letters to represent numbers • Record with algebraic symbols, for example, "John is *x* years old now. How old will he be after 10 years?" and record as (*x* + 10) years old	
Equations					Simple equations involving 1 step in finding the solution: • Understand the concept of equations • Solve simple equations involving 1 step in the solutions and check the answers (involving whole numbers only) • Solve problems by simple equations (involving only 1 step in the solutions)	Simple equations (involving 2 steps in finding the solution): • Solve equations involving at most 2 steps in the solutions, and examine the results • Solve problems by using simple equations (involving at most 2 steps in the solution)
NCTM Focal Points: Algebra						
Grade 6. Writing, interpreting, and using mathematical expressions and equations. Students write mathematical expressions and equations that correspond to given situations, they evaluate expressions, and they use expressions and formulas to solve problems. They understand that variables represent numbers whose exact values are						

not yet specified, and they use variables appropriately. Students understand that expressions in different forms can be equivalent, and they can rewrite an expression to represent a quantity in a different way (e.g., to make it more compact or to feature different information). Students know that the solutions of an equation are the values of the variables that make the equation true. They solve simple one-step equations by using number sense, properties of operations, and the idea of maintaining equality on both sides of an equation. They construct and analyze tables (e.g., to show quantities that are in equivalent ratios), and they use equations to describe simple relationships (such as 3x = y) shown in a table.